# [*r*,*s*,*t*]-Colouring of Paths, Cycles and Stars

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# Contents

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# Chapter 1

# Introduction

There are three groups of students in a Trainee School. Each one of them should have a one-day session with his coaching team, formed by all his teachers. How many days are needed in order to schedule all meetings?

| *Teachers:* | **Mr.Lopez** | **Mr.Perez** | **Mrs.Gonzalez** |
|---|---|---|---|
| | Mrs.Smith | Mrs.Smith | Mr.Key |
| *Groups:* | Mr.Key | Mrs.Grant | Mrs.Grant |
| | Mr.Nelson | Mr.Cox | Mr.Atkins |
| | | | Mrs.Lee |

Figure 1.1: The groups in the Trainee School

This question can be answered solving a vertex-colouring problem of the graph that models the situation: each student is represented by a vertex and two vertices are connected if and only if the corresponding students share a group, as follows.

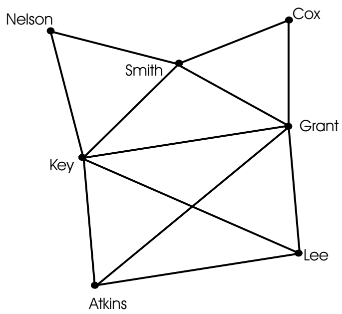
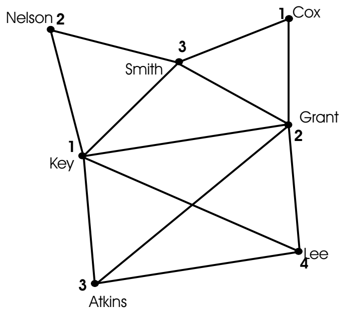



Figure 1.2: Graph representing the groups in the Trainee School

And then, colours are assigned to each vertex of the graph in such a way that adjacent vertices receive different colours. Hence, the graph will be proper coloured (in this thesis, we will always talk about proper colourings).
In this case the colour assigned to each vertex represents the day in which this

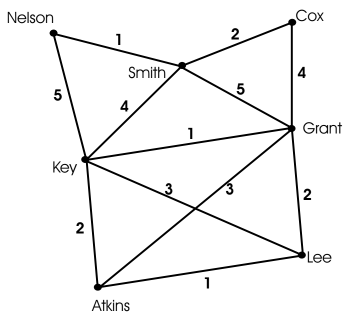
student is meeting his teachers. One possible solution is given by Figure 1.3




| | | |
|---|---|---|
| *Meetings 1st day:* | Key - **Lopez** and **Gonzalez** | Cox - **Perez** |
| *Meetings 2nd day:* | Nelson - **Lopez** | Grant - **Perez** and **Gonzalez** |
| *Meetings 3rd day:* | Smith - **Lopez** and **Perez** | Atkins - **Gonzalez** |
| *Meetings 4th day:* | Lee - **Gonzalez** | |

Figure 1.3: Vertex-Colouring

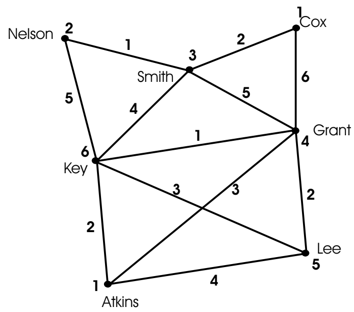
Hence, all meetings could be arranged in 4 days and obviously it could not be done in less days, since Mrs. Gonzalez has four students and she needs one day per each of them.
On the other hand, another possible situation is that the students should work in pairs with each of their group partners during a day to prepare a presentation about an hypothetic product. How many days are required in order to have all presentations done? This question can be solved by colouring the edges of the previous graph, where the colour assigned to an edge is the day when the corresponding pair is working together, for example as follows.




| | | | |
|---|---|---|---|
| *Meetings 1st day:* | Smith-Nelson | Key-Grant | Atkins-Lee |
| *Meetings 2nd day:* | Smith-Cox | Key-Atkins | Grant-Lee |
| *Meetings 3rd day:* | Key-Lee | Grant-Atkins | |
| *Meetings 4th day:* | Smith-Key | Cox-Grant | |
| *Meetings 5th day:* | Key-Nelson | Smith-Grant | |

Figure 1.4: Edge-Colouring

Hence, all presentations can be done in 5 days. It is an optimal solution, since Mr. Key and Mrs. Grant have five partners.
Following with these examples, one might think if both activities (the coaching sessions and the team work) can be scheduled in such a way that two activities cannot be done on the same day. Obviously it could be done in 9 days, just doing the coaching sessions at first and then prepare the presentations, but now, as we will see, this solution can be optimized if the activities can be alternated. It is a typical application of the total-colouring. One possible solution is shown in Figure 1.5 where, as before, the vertices colours represent the days in which this student is having his coaching session and the edges colours represent the days in which the corresponding students are meeting.




| | | | | |
|---|---|---|---|---|
| *Meetings 1st day:* | Smith-Nelson | Key-Grant | Cox-**Perez** | Atkins-**Gonzalez** |
| *Meetings 2nd day:* | Smith-Cox | Key-Atkins | Grant-Lee | Nelson-**Lopez** |
| *Meetings 3rd day:* | Key-Lee | Grant-Atkins | Smith-**Lopez** and **Perez** | |
| *Meetings 4th day:* | Smith-Key | Atkins-Lee | Grant-**Perez** and **Gonzalez** | |
| *Meetings 5th day:* | Key-Nelson | Smith-Grant | Lee-**Gonzalez** | |
| *Meetings 6th day:* | Grant-Cox | Key-**Lopez** and **Gonzalez** | | |

Figure 1.5: Total-Colouring

Hence, all activities can be done in 6 days. And this is an optimal timing since Mr. Key and Mrs. Grant need 5 days to prepare the presentations with their partners and one day for their coaching sessions.

As it has been shown, all these problems can be solved using classical colourings. But what happens if some other constraints are introduced? For example it would be natural that the teachers would need at least one day between two coaching sessions in order to prepare them. Or that the students do not need just one day to prepare the presentation, but that at least three.
This is a typical example in which the $[r, s, t]$-colouring can be used, in other words, a situation in which vertices and edges are coloured in such a way that elements of the graph that are in contact receive not only different colours, but there must be also a certain distance $r$ between colours of adjacent vertices, a distance $s$ between colours of adjacent edges and a distance $t$ between colours of incident vertices and edges (for a formal definition see Definition 2.21).

Then, the situation presented above can be considered as a $[2,3,1]$-colouring of the graph (if supposed that a coaching session cannot be done on the first day that one student is preparing a presentation), for example as shown in Figure 1.6.

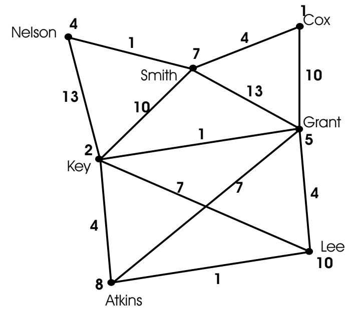



Figure 1.6: $[2,3,1]$-Colouring

Hence all activities with the demanded constraints can be achieved in 15 days (the presentations starting on the 13th day, still need two days to be done), which is optimal because Mr. Key and Mrs. Grant need for their five presentations at least this time.
On the other hand, some examples illustrate that this colouring respects the constraints: For instance, Mr.Key makes his 5 presentations, using three days for each of them, in such a way that he is starting the 5th presentation on the 13th day and he has his coaching session on the 2nd day, when he has not started preparing any presentation; on the other hand the teacher Mr.Lopez directs a coaching session with Mr.Key on the 2nd day, with Mr.Nelson on the 4th day and with Mrs.Smith on the 7th day, hence he always has at least 1 day to prepare the next coaching.

Another example of application of the $[r,s,t]$-colouring was introduced by Kemnitz and Marangio [11], as follows.

Assume that in a soccer tournament there is an elimination round where four teams are playing, such that each team plays one match against the others. During this round each team should get the possibility of a training day. Since there is only one training field, different training days must be assigned to the teams. Furthermore, a training day of a team should be different from a playing day and no team should play two successive days.

All required conditions are fulfilled in a $[1,2,1]$-colouring of a complete graph $K_4$ if one assigns the vertices of $K_4$ to the training days of the teams and the edges to the matches between them.

Figure 1.7 shows in the right picture a $[1,2,1]$-colouring with 6 colours of the complete graph $K_4$. It means then that one can arrange a schedule for the considered soccer tournament round fulfilling all the desired conditions in six days. On the other hand, the left picture shows a $[2,4,1]$-colouring of $K_4$ with 9 colours, which would model another situation in which two teams cannot train on two consecutive days, between the matches of a team there must be at least 3 free days and there

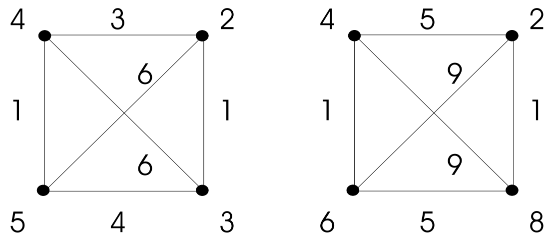



Figure 1.7: A $[2,4,1]$-colouring and a $[1,2,1]$-colouring of $K_4$

cannot be two activities for a team on the same day.

Hence, as seen, using $[r,s,t]$-colourings it is possible to solve more general situations than just with the classical colourings.

In Section 2.1 the classical colourings will be formally introduced and known results will be presented for some classes of graphs that are relevant for this thesis.

The notion of $[r,s,t]$-colouring was introduced by Arnfried Kemnitz, Massimiliano Marangio and Andrea Hackmann in a discussion in 2002 and a paper of them [11] is in preprint. There, the definition of $[r,s,t]$-colouring, certain properties and also several results for complete graphs are given. Some of them will be presented in Section 2.2.

As a first step to determine the $[r,s,t]$-chromatic number for some graphs, the simplest classes were studied. In Chapter 3 and Chapter 4 the exact value of the $[r,s,t]$-chromatic number for paths and cycles are determined, respectively.

Since for the $[r,s,t]$-colouring, the hereditary property for subgraphs holds (Lemma 2.22), one of the most interesting classes to be considered are the stars. Studying a graph $G$, the $[r,s,t]$-chromatic number of the star with $\Delta(G)$ (maximum degree of $G$) leaves is a lower bound for its $[r,s,t]$-chromatic number. In Chapter 5 the $[r,s,t]$-chromatic number for $K_{1,3}$ is determined and for $K_{1,n}$ bounds and some exact values are given.

Finally in Chapter 6 some more results for bipartite graphs and complete graphs are given.

The terminology and notation of the books by D.B. West [19] and H.P. Yap [20] will be mainly used. In addition a Glossary with some basic terminology and other terms that will be defined in this work can be found in the last pages.

# Chapter 2

# Preliminaries

## 2.1 Classical Colourings

One of the graph-theoretical parameters that has received more attention over the years is the chromatic number, undoubtedly due to its involvement in the Four Colour Theorem. It says that every map drawn on a sheet of paper can be coloured with just four colours in such a way that countries sharing a common border receive different colours (provided that all countries are connected regions).

The first known document on the Four Colour Problem is a letter dated October 23, 1852, written by August DeMorgan to his colleague Sir William Rowan Hamilton. There, he wrote that his student Frederick Guthrie (who later attributed it to his brother Francis) asked him about the Four Colour Problem and that he was not able to solve it.

In 1879, one year after the proposal of the problem to the London Mathematical Society by Arthur Cayley, Alfred Bray Kempe published a paper [12] that claimed to prove that the conjecture was true. In 1890, Percy J. Heawood published a refutation and a first proof of the Five Colour Theorem [8].

Even though, Kempe's argument contained most of the basic ideas that eventually lead to the correct proof by Kenneth Appel and Wolfgang Haken working with John Koch [1, 2, 3] one century later, in 1976. It is a quite unusual proof, due to the fact that it made unprecedented use of computer computation: the correctness of the proof cannot be checked without the aid of computer and some of the crucial ideas were upgraded by computer experiments.

In 1997, Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas gave a simpler proof of the Four Colour Theorem [15, 16], but which still contains several hundred irreducible cases.

The Four Colour Problem could be seen as a vertex-colouring problem for a planar graph being the dual graph of a map $G$. The dual graph $G^*$ of a map $G$ is a planar graph whose vertices correspond to the faces of $G$ and the edges correspond to the edges of $G$ as follows: if $e$ is an edge of $G$ with face $X$ on one side and face $Y$ on the other side, then the end points of the dual edge $e^* \in E(G^*)$ are the vertices $x, y \in G^*$ that represent the faces $X, Y$ of $G$.

### 2.1.1 Vertex-Colouring

The colouring of the faces of a map corresponds to a colouring of the vertices of its line graph defined as follows.

**Definition 2.1.** A proper *k-colouring* (or $k$-vertex-colouring) of a graph $G$ is a labelling $f : V(G) \to S$, where $|S| = k$ and adjacent vertices have different labels. The labels are called *colours*. A graph is *k-colourable* if it has a proper $k$-colouring. The *chromatic number* $\mathcal{X}(G)$ is the least $k$ such that $G$ is $k$-colourable.

If $f$ is a proper $k$-colouring of $G$, then $f$ yields a partition of the set of vertices of $G$, $V(G)$, into independent sets $V_1, \ldots, V_k$. These independent sets $V_1, \ldots, V_k$ of vertices of $G$ are called *colour classes* of $f$.

And the Four Colour Theorem for maps leads to the corresponding theorem for graphs.

**Theorem 2.2 (Four Colour Theorem).** *For any simple connected planar graph exists a proper* 4*-colouring.*

Putting aside planar graphs and the Four Colour Theorem, there will be shown some general known results for vertex-colouring of graphs.

Graphs with loops are uncolourable, since the colour of a vertex cannot be made different from itself. Therefore, in this subsection all graphs are loopless. Also, multiple edges are irrelevant, because extra copies of edges do not affect colourings. Thus it will be thought in terms of simple graphs. Most of the statements made without restriction to simple graphs remain valid when multiple edges are allowed. The investigation can be also restricted to connected graphs, since the number of colours needed to colour a disconnected graph is the maximum of the colours needed to colour each of its components.

Although the chromatic number is one of the most studied parameters in graph theory, no practicable formula exists for the chromatic number of an arbitrary graph. In fact, Stockmeyer [14] proved that the 3-colorability problem is NP-complete, which leads to the NP-completeness of the $k$-colorability problem for $k \geq 3$. This means that there is no known polynomial-time algorithm that can answer the question whether a graph is $k$-colorable or not.

The goal is to find an upper and a lower bound that coincide, because then $\mathcal{X}(G)$ is equal to this common value. But in fact, in many cases there will only be possible to find bounds close enough one from the other.

If the clique number, $\omega(G)$, of a graph $G$ is the maximum size of a set of pairwise adjacent vertices (clique) in $G$, then an obvious bound is $\mathcal{X}(G) \geq \omega(G)$, since in the

clique each vertex must receive a different colour.
This idea yields to the concept of perfect graph, a graph in which this bound is sharp, so $\mathcal{X}(G) = \omega(G)$. In 1960, Claude Berge [7] conjectured that, every graph with no odd hole or antihole is perfect, where an odd hole is an induced subgraph which is an odd cycle of length at least five, and an odd antihole is the same in the complement graph.
A proof of the conjecture was not completed until 42 years later. It was 2002 when Paul Seymour, Maria Chudnovsky, Neil Robertson and Robin Thomas finally completed it.

On the other hand, an upper bound for the chromatic number of graphs can be easily found. By induction, it is proved that a graph $G$ with maximum vertex degree $\Delta(G)$ can be coloured with $\Delta(G) + 1$ colours. This bound was improved by Brooks [6].

**Theorem 2.3 (Brooks' Theorem).** *Let $G$ be a connected simple graph. If $G$ is neither a cycle with an odd number of vertices, nor a complete graph, then $\mathcal{X}(G) \leq \Delta(G)$.*

### Vertex-Colouring for some Classes of Graphs

The bound mentioned before the Brook's Theorem, $\mathcal{X}(G) \leq \Delta(G) + 1$, holds with equality for complete graphs and odd cycles.

**Lemma 2.4.** *For the complete graph with $n$ vertices, $K_n$, it holds*

$$\mathcal{X}(G) = \Delta(K_n) + 1 = n,$$

*for all $n$.*

*Proof.* Since any vertex has $\Delta(K_n) = \Delta$ neighbors, obviously $\Delta + 1$ colours are at least needed, which coincides with the upper bound. □

**Lemma 2.5.** *For any cycle of odd order, $C_{2n+1}$, it holds*

$$\mathcal{X}(C_{2n+1}) = \Delta(C_{2n+1}) + 1 = 3.$$

*Proof.* Since each colour class is an independent set, a graph is 2-colourable if and only if it is bipartite. Then, odd cycles, which are no bipartite graphs, have chromatic number greater or equal 3 and since they are 3-colourable $\mathcal{X}(C_{2n+1}) = 3$ for all $n \geq 1$. □

In the previous proof there was a very interesting result for this thesis, which follows from the fact that each colour class is an independent set.

**Lemma 2.6.** *For any bipartite graph $G$, it holds*

$$\mathcal{X}(G) = 2.$$

Hence, this value holds for non trivial paths, cycles of even order and stars, since all of them are bipartite graphs.

### 2.1.2 Edge-Colouring

Many of the previous questions and results are naturally analogues for edges and this can be explained by means of the relation between independent sets and matchings: independent sets have no adjacent vertices and matchings have no adjacent edges. Then, if vertex-colourings partition the vertex set into independent sets, in an analogous way the edge set can be partitioned into matchings.

**Definition 2.7.** A proper *$k$-edge-colouring* of a graph $G$ is a labelling $f : E(G) \to S$, where $|S| = k$ and adjacent edges have different labels.
Again the labels are called *colours*. A graph is *$k$-edge-colourable* if it has a proper $k$-edge-colouring. The *edge-chromatic number* (or chromatic index) $\mathcal{X}'(G)$ is the least $k$ such that $G$ is $k$-edge-colourable.

If $f$ is a proper $k$-edge-colouring of $G$, then $f$ yields a partition of $E(G)$ into independent sets $E_1, \ldots, E_k$. These independent sets $E_1, \ldots, E_k$ of edges of $G$ are called *colour classes* of $f$.

Edge-colouring and vertex-colouring are related via line graphs.

**Definition 2.8.** The *line graph* of $G$, written $L(G)$, is the simple graph whose vertices are the edges of $G$, with $ef \in E(L(G))$ when $e$ and $f$ are adjacent in $G$.

Then, questions about edges in a graph $G$ can be phrased as questions about vertices in $L(G)$. This observation appears to be of little value in computing edge-chromatic number, since chromatic numbers are in general extremely difficult to evaluate. If the chromatic-number can be computed, then the original question about edges in $G$ can be answered by applying the vertex result to $L(G)$.
However, note that from any graph $G$, $L(G)$ can be build, but not for any graph $H$, a graph $G$, such that $L(G) = H$, can be found.

In contrast with Subsection 2.1.1, multiple edges greatly affect $\mathcal{X}'(G)$, but in this thesis just simple graphs will be considered. A graph with a loop has no proper edge-colouring, thus it will be thought in terms of loopless graphs. As on vertex-colouring, the study can be restricted to connected graphs, since the number of colours needed to edge-colour a disconnected graph is the maximum of the colours needed to edge-colour each of its components.

Since edges sharing a vertex need different colours, $\mathcal{X}'(G) \geq \Delta(G)$. The upper bound $\mathcal{X}'(G) \leq 2\Delta(G) - 1$ also follows easily: if the edges are coloured in some order, always assigning to the current edge the least-indexed colour different from those already appearing on edges adjacent to it. Since no edge is adjacent to more than $2(\Delta(G) - 1)$ other edges, this never uses more that $2\Delta(G) - 1$ colours.
Using a procedure in which $\Delta(G) + 1$ colours are available and a proper edge-colouring is built, incorporating edges one by one until a proper $\Delta(G) + 1$-edge-colouring is found, much better upper bounds have been established by Vizing [17, 18]. This shows that the trivial lower bound is almost sharp.

**Theorem 2.9 (Vizing's Theorem).** *Let $G$ be a simple graph, then* $\Delta(G) \leq \mathcal{X}'(G) \leq \Delta(G) + 1$.

Hence, all graphs can be divided in two classes.

**Definition 2.10.** A simple graph $G$ is *Class 1* if $\mathcal{X}'(G) = \Delta(G)$ and it is *Class 2* if $\mathcal{X}'(G) = \Delta(G) + 1$.

Anyway, determining whether a graph is Class 1 or Class 2 is, as Holyer [9] proved, generally an NP-complete problem.

### Edge-Colouring for some Classes of Graphs

For bipartite graphs the trivial lower bound is achieved and there is a good algorithm to obtain a proper $\Delta(G)$-edge-colouring in a bipartite graph $G$, as König [13] proved.

**Theorem 2.11 (König's Theorem).** *All bipartite graph are Class 1.*

Thus paths, cycles of even order and stars are Class 1.
On the other hand, cycles of odd order are Class 2.

**Lemma 2.12.** *All odd cycles, $C_{2n+1}$, are Class 2.*

*Proof.* For a regular graph $G$, a proper edge-colouring with $\Delta(G)$ colours is equivalent to a decomposition into perfect matchings. Since odd cycles are regular and have no such a decomposition, $\mathcal{X}'(C_{2n+1}) > \Delta(C_{2n+1}) = 2$, and $C_{2n+1}$ is 3-edge-colourable, so $\mathcal{X}'(C_{2n+1}) = 3$, for all $n$. $\square$

The chromatic index of complete graphs has been studied by many authors, for example Vizing [18], Behzad, Chartrand and Cooper [5].

**Lemma 2.13.** *For all complete graphs, $K_n$, it holds*

$$\mathcal{X}'(K_n) = \begin{cases} n-1 & \text{if } n \text{ even;} \\ n & \text{if } n \text{ odd.} \end{cases}$$

### 2.1.3 Total-Colouring

The notion of total-colouring is a generalization of the previous concepts and it was introduced and studied by Behzad [4] and Vizing [17] around the year 1965. With this colouring, not just vertices or edges have to be coloured, but every element of the graph (every element of $V(G) \cup E(G)$) has to be coloured in such a way that neighboring elements receive different colour.

**Definition 2.14.** A proper *k-total-colouring* of a graph $G$ is a labelling $f : V(G) \cup E(G) \to S$, where $|S| = k$ and no two adjacent vertices or edges have the same label and the image of each vertex is distinct from the images of its incident edges.
The labels are again called *colours*. A graph is *k-total-colourable* if it has a proper $k$-total-colouring. The *total-chromatic number* $\mathcal{X}_T(G)$ is the least $k$ such that $G$ is $k$-total-colourable.

Thus if $f$ is a total-colouring of $G$, then $f|_{V(G)}$, the restriction of $f$ on $V(G)$, is a vertex-colouring of $G$. Similarly, $f|_{E(G)}$ is an edge-colouring of $G$. Taking this into account, it follows that a total-colouring $f$ of $G$ yields a partition of $V(G) \cup E(G)$ into independent sets $V_1 \cup E_1,\ V_2 \cup E_2, \dots,$ where $V_1, V_2, \dots$ are independent sets of vertices of $G$ and $E_1, E_2, \dots$ are independent sets of edges of $G$, and no vertex in $V_i$ is incident with any edge in $E_i$. These independent sets $V_1 \cup E_1,\ V_2 \cup E_2, \dots$ are called the colour classes of $f$. Conversely, any partition of $V(G) \cup E(G)$ into independent sets $V_1 \cup E_1,\ V_2 \cup E_2, \dots, V_k \cup E_k$ leads to a k-total-colouring of $G$.

Since a total-colouring of $G$ is a vertex-colouring and an edge-colouring of $G$ at the same time, the degree of difficulty of this subject is obvious.

Similar to the study of vertex- and edge-colouring of graphs, in the study of the total-colouring of a graph $G$, it shall be assumed that $G$ is connected. On the other hand, it will be also restricted to loopless and simple graphs.

Clearly, for any graph $G$, $\mathcal{X}_T(G) \geq \Delta(G) + 1$, since a vertex of maximum degree needs a different colour from those $\Delta(G)$ assigned to its incident edges.

On the other hand, since every element of the graph $G$ (every element of $V(G) \cup E(G)$) has at most $2\Delta(G)$ neighbors, a trivial upper bound is $\mathcal{X}_T(G) \leq 2\Delta(G) + 1$, for all graphs $G$.
The following conjecture aiming a better general upper bound was posed independently by Behzad [4] and Vizing [17] in 1965.

**Theorem 2.15 (Total-Colouring Conjecture (TCC)).** *For any graph $G$, $\mathcal{X}_T \leq \Delta(G) + 2$.*
*(In fact, Vizing posed a more general conjecture for graphs with multiple edges which says that for any multigraph $G$, $\mathcal{X}_T \leq \Delta(G) + \mu(G) + 1$, where $\mu(G)$ denotes the maximum multiplicity of edges in $G$.)*

Hence if the TCC is considered true, all simple graphs can be divided in two classes.

**Definition 2.16.** A simple graph $G$ is *Type 1* if $\mathcal{X}_T(G) = \Delta(G) + 1$ and it is *Type 2* if $\mathcal{X}_T(G) = \Delta(G) + 2$.

The TCC was proved true for a few classes of graphs in the 1970's. Only recently, some new techniques have been introduced and used to prove that the TCC holds for some more classes of graphs, especially graphs having high maximum degree.

For example, the TCC holds for bipartite graphs: since $\mathcal{X}(G) = 2$ (Lemma 2.6) and $\mathcal{X}'(G) = \Delta(G)$ (König's Theorem 2.11) for all bipartite graph $G$, vertices and edges can be coloured separately using no more that $\Delta(G)+2$ colours. On the other hand, Behzad, Chartrand and Cooper [5] proved the TCC for cycles in 1971.

### Total-Colouring for some Classes of Graphs

For paths and cycles, the total-chromatic number can be easily determined: first, search for a possible colouring with $\Delta(G)+1$ colours. If it not possible, then $\Delta(G)+2$ colours are enough, since the TCC holds for them.

**Lemma 2.17.** *Every path, $P_n$, is Type 1.*

**Lemma 2.18.** *For every cycle, $C_n$, it holds*

$$\mathcal{X}_T(C_n) = \begin{cases} 3 & \text{if } \; n \equiv 0(mod\ 3); \\ 4 & \text{if } \; n \not\equiv 0(mod\ 3). \end{cases}$$

Finally, the total-chromatic number for complete bipartite graphs (in particular for stars) and for complete graphs was determined by Behzad, Chartrand and Cooper [5] in 1967.

**Lemma 2.19.** *For any complete bipartite graph, $K_{n,m}$, it holds*

$$\mathcal{X}_T(K_{n,m}) = \begin{cases} \Delta(K_{n,m}) + 1 & \text{if } \; n \neq m; \\ \Delta(K_{n,m}) & \text{if } \; n = m. \end{cases}$$

**Lemma 2.20.** *For any complete graph, $K_n$, it holds*

$$\mathcal{X}_T(K_n) = \begin{cases} \Delta(K_n) + 1 = n & \text{if } \; n \text{ odd;} \\ \Delta(K_n) = n - 1 & \text{if } \; n \text{ even.} \end{cases}$$

## 2.2 $[r, s, t]$-Colouring

In 2002, Hackman, Kemnitz and Marangio, working on total-colourings, observed that some situations could not be modelled with this colouring and introduced a new concept: the $[r, s, t]$-colouring. As before, vertices and edges have to be coloured, but neighboring elements have to receive not only different colours but also colours with a certain difference $r$ between colours of adjacent vertices, a distance $s$ between colours of adjacent edges and a distance $t$ between colours of incident vertices and edges. Observe that these distances can be defined because the labels, that we called colours, are in fact natural numbers.

**Definition 2.21.** Given non-negative integers $r$, $s$ and $t$, an $[r, s, t]$-*colouring* of a graph $G = (V(G), E(G))$ is a mapping $c$ from $V \cup E(G)$ to the colour set $\{1, 2, ..., k\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices $v_i, v_j$, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges $e_i, e_j$, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices $v_i$ and edges $e_j$, respectively.
The $[r, s, t]$-*chromatic number* $\mathcal{X}_{r,s,t}(G)$ of $G$ is defined to be the minimum $k$ such that $G$ admits an $[r, s, t]$-colouring.

Obviously, a $[1, 0, 0]$-colouring is a classical vertex-colouring, a $[0, 1, 0]$-colouring is a classical edge-colouring and a $[1, 1, 1]$-colouring is a classical total-colouring.

Hence, there are several different applications for such $[r, s, t]$-colourings.

### Properties of the $[r, s, t]$-Chromatic Number

Some hereditary properties of the $[r, s, t]$-chromatic number were given by Kemnitz and Marangio [11].
For instance, since the restriction of an $[r, s, t]$-colouring of a graph, $G$, to the elements of any of its subgraphs, $H \subseteq G$, is an $[r, s, t]$-colouring of $H$, the $[r, s, t]$-chromatic number of any subgraph of $G$ is a lower bound for the $[r, s, t]$-chromatic number of $G$.

**Lemma 2.22.** *If $H \subseteq G$ then*

$$\mathcal{X}_{r,s,t}(H) \leq \mathcal{X}_{r,s,t}(G).$$

On the other hand, an $[r, s, t]$-colouring of $G$ is, by definition, an $[r', s', t']$-colouring for $G$ for any $r'$, $s'$ and $t'$, such that $r' \leq r$, $s' \leq s$ and $t' \leq t$.

**Lemma 2.23.** *If $r' \leq r$, $s' \leq s$, $t' \leq t$ then*

$$\mathcal{X}_{r',s',t'}(G) \leq \mathcal{X}_{r,s,t}(G).$$

Using a vertex-colouring (or an edge-colouring) of the graph $G$, an $[r, 0, 0]$-colouring (or a $[0, s, 0]$-colouring) of $G$ can be easily found. The vertices (or edges) have to be recoloured in such a way that an element, that had the colour $i$ in the original vertex-colouring (or edge-colouring), receive the colour $r(i - 1) + 1$ (or $s(i - 1) + 1$).

**Lemma 2.24.** *If $G$ is non-trivial then*

1. $\mathcal{X}_{r,0,0}(G) = r(\mathcal{X}(G) - 1) + 1$,
2. $\mathcal{X}_{0,s,0}(G) = s(\mathcal{X}'(G) - 1) + 1$.

Kemnitz and Marangio gave also bounds for the $[r, s, t]$-chromatic number of a general graph $G$, as follows

$$\max\{r(\mathcal{X}(G)-1)+1, s(\mathcal{X}'(G)-1)+1, t+1\} \leq \mathcal{X}_{r,s,t}(G) \leq r(\mathcal{X}(G)-1)+s(\mathcal{X}'(G)-1)+t+1,$$

which can be improved in a first step as shown in Lemma 2.25.

**Lemma 2.25.** *For the $[r, s, t]$-chromatic number of a graph $G$, it holds*

$$\max\{r(\mathcal{X}(G)-1)+1, s(\mathcal{X}'(G)-1)+1, t+\Delta(G)\} \leq \mathcal{X}_{r,s,t}(G) \leq r(\mathcal{X}(G)-1)+s(\mathcal{X}'(G)-1)+t+1,$$

*if $|V(G)| \geq 2$, $G \neq \overline{K_n}$ and $s, t \geq 1$ (where $\Delta(G)$ is the maximum degree of $G$ and $\overline{K_n}$ is the empty graph).*

*Proof.* By Lemma 2.24 and 2.23
$r(\mathcal{X}(G) - 1) + 1 = \mathcal{X}_{r,0,0}(G) \leq \mathcal{X}_{r,s,t}(G)$
$s(\mathcal{X}'(G) - 1) + 1 = \mathcal{X}_{0,s,0}(G) \leq \mathcal{X}_{r,s,t}(G)$

On the other hand, the star $K_{1,\Delta}$ consisting of a vertex of maximum degree $\Delta := \Delta(G)$, $v_0$, and its adjacent vertices without other edges than the ones connecting $v_0$ and its neighbors, needs at least the following number of colours: To colour the $\Delta$ edges at least $s(\Delta - 1) + 1$ colours have to be used, which is at least $\Delta$. The colour of the "central vertex" can be set as follows.

If this colour is smaller than the smallest one for the edges or greater than the greatest, at least $t + \Delta$ colours are being used. If the colour for the vertex "fits" between the colours of two edges, the difference between these two colours must be at least $2t$, hence the total number of colours used to colour the edges is at least $2t + \Delta - 1$, which is at least $t + \Delta$, if $t \geq 1$.

Then, by Lemma 2.22, $t + \Delta \leq \mathcal{X}_{r,s,t}(K_{1,\Delta}) \leq \mathcal{X}_{r,s,t}(G)$. Hence, $\max\{r(\mathcal{X}(G) - 1) + 1, s(\mathcal{X}'(G) - 1) + 1, t + \Delta\} \leq \mathcal{X}_{r,s,t}(G)$. Observe that, if $s = 0$, $t + \Delta$ would not be a lower bound but $t + 1$; for the other bounds nothing changes.

For the upper bound, it is enough to find a proper $[r, s, t]$-colouring with the desired number of colours. If the vertices of $G$ are coloured (see [11]) with colours $0, r, \ldots, r(\mathcal{X}(G) - 1)$ and the edges with colours $r(\mathcal{X}(G) - 1) + t, r(\mathcal{X}(G) - 1) + t + s, \ldots, r(\mathcal{X}(G) - 1) + t + s(\mathcal{X}'(G) - 1)$, then an $[r, s, t]$-colouring of $G$ is obtained. □

The previous Lemmas 2.22 and 2.25 will be used to determine bounds for the $[r, s, t]$-chromatic number.

### Sharpness of the lower bounds

In this subsection the sharpness of the lower bounds given in Lemma 2.25 will be proved.

**Lemma 2.26.** *For any graph $G$,*
*if $r \geq \lceil \frac{\Delta(G)}{\mathcal{X}(G)-1} \rceil s + 2t$, or $r \geq \lceil \frac{\Delta(G)+2-\mathcal{X}(G)}{\mathcal{X}(G)-1} \rceil s + 2t$ and $s < 2t$,*
*then $\mathcal{X}_{r,s,t}(G) = (\mathcal{X}(G) - 1)r + 1$,*
*and if $s \geq r + 2t$ and $r < 2t$ or $2s \geq 3r + 2t$,*
*then $\mathcal{X}_{r,s,t}(G) = (\mathcal{X}'(G) - 1)s + 1$.*

*Proof.* (1.1) If $r \geq \lceil \frac{\Delta(G)}{\mathcal{X}(G)-1} \rceil s + 2t$, then the elements of the graph can be coloured as follows:
The vertices are coloured with the $\mathcal{X}(G)$ colours $1, r+1, 2r+1, \dots, (\mathcal{X}(G)-1)r+1$. And the edges use the following colours: $a_1 = t+1$, $a_{j+1} = a_j + s$, for all $j \geq 1$ and $j \neq nk+1$ and $a_{nk+2} = \max\{a_{nk+1} + s, nr + t + 1\}$, for all $n$, where $k := \lceil \frac{\Delta(G)}{\mathcal{X}(G)-1} \rceil$.
In this way the colours of the edges were placed in the intervals between the colours of the vertices. Then, in order not to use more than $(\mathcal{X}(G)-1)r+1$ colours, the edges can receive $k+1+k(\mathcal{X}(G)-2) \geq \Delta + 1 \geq \mathcal{X}'(G)$ different colours, where the inequality $\mathcal{X}'(G) \leq \Delta + 1$ follows from the Vizing's Theorem (see Theorem 2.9).

(1.2) If $s < 2t$ and $r \geq \lceil \frac{\Delta(G)+2-\mathcal{X}(G)}{\mathcal{X}(G)-1} \rceil s + 2t$, then the vertices can receive colours from the list $1, r+1, \dots, (\mathcal{X}(G)-1)r+1$ and the edges from the following $a_i$, defined as $a_1 = t+1$, $a_{i+1} = a_i + s$ for $i \neq n(k+1)-1$ for some $n$, and $a_{n(k+1)} = nr + t + 1$ for all n, where $k := \lceil \frac{\Delta(G)+2-\mathcal{X}(G)}{\mathcal{X}(G)-1} \rceil$.
Then, similarly as above, among the $(\mathcal{X}(G)-1)(r-1)$ remaining colours smaller than $(\mathcal{X}(G)-1)r+1$, there are $(k+1)(\mathcal{X}(G)-1) \geq \Delta + 1$ possible colours for the edges, which are enough.

(2) If $s \geq r + 2t$, then the elements of the graph can be coloured using the following colours: For the edges, the colours $1, s+1, \dots, (\mathcal{X}'(G)-1)s+1$ are chosen (observe that, in this way, the remaining colours smaller than $(\mathcal{X}'(G)-1)s+1$ are divided into $\mathcal{X}'(G)-1$ intervals, each containing $s-1$ colours). For the vertices we use the colours $t+1, r+t+1$, which fit in the first interval of colours; $\max\{s, 2r\} + t + 1, \max\{s+r, 3r\} + t + 1$ that are in the second interval (because if $r < 2t$, then $2s+1 \geq s+r+2t+1 > 3r+2t+1$ and in the other case if $2s \geq 3r+2t$, then $2s+1 \geq \max\{s+r, 3r\} + 2t + 1$); and finally $\max\{2s, 4r\} + t + 1, \max\{3s, 5r\} + t + 1, \dots, \max\{(\mathcal{X}'(G)-2)s, \mathcal{X}'(G)r\} + t + 1$, where each one lays in one of the following intervals.
In this way, there are $\mathcal{X}'(G)+1$ possible colours for the vertices, which is greater or equal to $\Delta + 1$ and hence greater or equal to $\mathcal{X}(G)$, where the last inequality is a consequence of the Brooks' Theorem (see Theorem 2.3).

□

On the other hand, the third lower bound is achieved in a not so general case, which is natural, since it can still be improved as shown in Lemma 5.1.

**Lemma 2.27.** *For a star with $n$ leaves $K_{1,n}$, $\mathcal{X}_{1,1,t}(K_{1,n}) = t + \Delta(K_{1,n})$ if $t < n = \Delta(K_{1,n})$.*

*Proof.* Kemnitz and Marangio [11] proved that for any bipartite graph $G$, $t + \Delta(G) \leq \mathcal{X}_{1,1,t}(G) \leq t + \Delta(G) + 1$. Then, if $v_0$ is the central vertex of the star, the $n$ edges are noted as $e_1, e_2, \dots, e_n$ and $v_i$ is the leaf adjacent to $e_i$, $K_{1,n}$ could be coloured as follows: $c(v_0) = 1$, $c(v_2) = 2t+1$, $c(v_i) = 2$ for all $i = 2 \dots n$ and $c(e_j) = j+1$ for all $j = 1 \dots n$. Which is a $[1,1,t]$-colouring with $t+n$ colours, hence $\mathcal{X}_{1,1,t}(K_{1,n}) = t + \Delta(K_{1,n})$. □

After this Lemma, a natural next step is to see what happens in the other case, this is if $t \geq \Delta(K_{1,n})$. Giving an answer to this question a better lower bound for bipartite graphs can be found. This will be done using the result given by Kemnitz and Marangio mentioned in the previous proof.

**Theorem 2.28.** *For any bipartite graph $G$, $\mathcal{X}_{r,s,t}(G) \geq t + \Delta(G) + 1$, if $t \geq \Delta(G)$.*

*Proof.* In a personal communication with Kemnitz and Marangio, they presented the proof of $\mathcal{X}_{1,1,t}(K_{1,\Delta(G)}) = \Delta(G) + t + 1$ if $t > \Delta$, which can be done as follows. It will be proved that if $t \geq \Delta$, then there is no $[1, 1, t]$ colouring of $K_{1,\Delta}$ with $t + \Delta$ colours.
Suppose $\mathcal{X}_{1,1,t}(K_{1,\Delta}) \leq t + \Delta$. If $c(v_0) < c(e_i) < c(v_i)$ (or the symmetric situation) for some $i$, where $v_0$ is the central vertex and $e_i$ is incident to $v_0$ and $v_i$, then $c(v_i)$ (or $c(v_0)$) $\geq 2t + 1 > t + \Delta$, which is a contradiction.
Then, $c(v_0), c(v_i) < c(e_i)$ (or the symmetric situation) for all $i$. Since $s = 1$, all edges should receive different colours, so the edge with the smallest colour can be noted as $e_1$ and so on until $e_\Delta$. Then $c(e_1) \geq t + 2$ and $c(v_\Delta) \geq t + \Delta + 1$, a contradiction.
Hence, $\mathcal{X}_{1,1,t}(K_{1,\Delta}) = t + \Delta + 1$.
Then, by Lemma 2.22 and Lemma 2.23, $\mathcal{X}_{r,s,t}(G) \geq \mathcal{X}_{r,s,t}(K_{1,\Delta}) \geq \mathcal{X}_{1,1,t}(K_{1,\Delta})$ □

### Treatment of symmetric cases

The following Lemma will be useful to reduce the number of cases to be considered.

**Lemma 2.29.** *If $c$ is an $[r, s, t]$-colouring with colours from $\{1, ..., k\}$, then $c'$ with $c'(x) = k + 1 - c(x)$ is also an $[r, s, t]$-colouring with colours from $\{1, ..., k\}$.*

*Proof.* If $1 \leq c(x) \leq k$, then $1 \leq k + 1 - c(x) \leq k$. And if $|c(x) - c(y)| \geq d$, then $|(k + 1 - c(x)) - (k + 1 - c(y))| = |c(y) - c(x)| \geq d$. □

In [11], Kemnitz and Marangio presented first results on $\mathcal{X}_{r,s,t}(G)$ such as general bounds and also exact values, for example for complete graphs and in the case $\min\{r, s, t\} = 0$.

# Chapter 3

# $[r, s, t]$-Colouring of Paths

As a first approach to the problem of the $[r, s, t]$-colouring of graphs, the $[r, s, t]$-colouring of paths was studied. Not just because of its expected easiness, but also because it could be used, because of Lemma 2.22, as a lower bound for graphs containing a certain path as a subgraph.

Since any path with $n$ vertices $P_n$ is bipartite, $\mathcal{X}(P_n) = 2$ for all $n$ and $\mathcal{X}'(P_n) = \Delta = 2$ for all $n \geq 3$ (for $n = 2$, $\mathcal{X}'(P_n) = \Delta = 1$), as observed in Lemma 2.6 and Theorem 2.11. Then these values can be substituted in Lemma 2.25 and some bounds for $\mathcal{X}_{r,s,t}(P_n)$ can be found.

**Corollary 3.1.** *For any path $P_n$, from Lemma 2.25 it follows that*

$$\max\{r+1, s+1, t+2\} \leq \mathcal{X}_{r,s,t}(P_n) \leq r+s+t+1, \textit{ for all } n \geq 3.$$

*(for $n = 2$, it would be* $\max\{r+1, t+2\} \leq \mathcal{X}_{r,s,t}(P_n) \leq r+t+1$*).*

**Notation 1.** *In this chapter there will be used the following notation for the colouring of the vertices and edges of a path*

$$(..., c(e_0), \mathbf{c(v_0)}, c(e_1), \mathbf{c(v_1)}, c(e_2), \mathbf{c(v_2)}, c(e_3), ...),$$

*where $\ldots, v_0, v_1, \ldots$ are vertices and $\ldots, e_0, e_1, \ldots$ edges of the considered path, such that $e_i = v_{i-1}v_i$.*

## 3.1 Some Lower Bounds for $\mathcal{X}_{r,s,t}(P_n)$

### 3.1.1 Lower Bounds for $\mathcal{X}_{r,s,t}(P_2)$ and $\mathcal{X}_{r,s,t}(P_3)$

As a first step on the study of the $[r, s, t]$-colouring for paths, some lower bounds will be given for the paths of order 2 and 3, since its $[r, s, t]$-chromatic number is a lower bound for the $[r, s, t]$-chromatic number of any path with greater order, because of Lemma 2.22.

**Observation 3.1.** *There is considered a path $P_2$ given by $v_0e_1v_1$. It may be assumed $c(v_0) \leq c(v_1)$. Then, by Lemma 2.29, all possible constellations of colours of its elements can be reduced to the following two:*

*If $c(v_0) \leq c(v_1) \leq c(e_1)$, then $k \geq r+t+1$.*
*If $c(v_0) \leq c(e_1) \leq c(v_1)$, then $k \geq \max\{2t+1, r+1\}$,*

*where $k := \mathcal{X}_{r,s,t}(G)$ (this notation will be used all along this thesis).*

*Hence*

$$k \geq \min\{r+t+1, \max\{2t+1, r+1\}\} = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

*Consequently for any path of order $n \geq 2$, by Lemma 2.22, these lower bounds are valid.*

**Observation 3.2.** *There is considered a path $P_3$ given by $v_0e_1v_1e_2v_2$. By symmetry, it can be assumed that $c(e_1) \leq c(e_2)$. Three main cases are now distinguished, which due to Lemma 2.29 can be reduced to two cases, and several subcases:*

*(Observe that, by Corollary 3.1, it may be assumed $k \leq r+s+t+1$. Hence the cases for which $k \geq k_0 > r+s+t+1$ can be omitted.)*

**1.** *If $c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq s+t+1$.*

*1.1 If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.*
*For this constellation, there are 5 possible situations, corresponding to the 5 possible relations between the colour of $v_2$ and the colours of the other elements of $P_3$.*

*If $c(v_2) \leq c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.*
*If $c(v_0) \leq c(v_2) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.*
*If $c(v_0) \leq c(e_1) \leq c(v_2) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+t+1, s+2t+1, 3t+1\}$.*
*If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_2) \leq c(v_1)$, then $k \geq r+s+2t+1 > r+s+t+1$.*
*If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1) \leq c(v_2)$, then $k \geq r+s+2t+1 > r+s+t+1$.*

*In the same way, the following cases are treated using shortened tables.*

*1.2 If $c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+t+1, s+t+1, 2t+1\}$.*

| | |
|---|---|
| $c(v_2) \leq c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1)$ | $k \geq \max\{r+t+1, s+t+1, 2t+1\}$ |
| $c(e_1) \leq c(v_2) \leq c(v_0) \leq c(e_2) \leq c(v_1)$ | |
| $c(e_1) \leq c(v_0) \leq c(v_2) \leq c(e_2) \leq c(v_1)$ | $k \geq \max\{r+t+1, s+t+1, 3t+1\}$ |
| $c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_2) \leq c(v_1)$ | $k \geq r+s+t+1$ |
| $c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1) \leq c(v_2)$ | |

*1.3 If* $c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1)$, *then* $k \geq \max\{r+t+1, s+t+1, r+s+1\}$.

| | |
|---|---|
| $c(v_2) \leq c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1)$ | $k \geq \max\{r+t+1, s+t+1, r+s+1, 2t+1\}$ |
| $c(e_1) \leq c(v_2) \leq c(e_2) \leq c(v_0) \leq c(v_1)$ | |
| $c(e_1) \leq c(e_2) \leq c(v_2) \leq c(v_0) \leq c(v_1)$ | $k \geq r+s+t+1$ |
| $c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_2) \leq c(v_1)$ | |
| $c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1) \leq c(v_2)$ | |

*1.4 If* $c(e_1) \leq c(e_2) \leq c(v_1) \leq c(v_0)$, *then* $k \geq r+s+t+1$.

**2.** *If* $c(e_1) \leq c(v_1) \leq c(e_2)$, *then* $k \geq 2t+1$.

*2.1 If* $c(v_0) \leq c(e_1) \leq c(v_1) \leq c(e_2)$, *then* $k \geq \max\{r+t+1, s+t+1, 3t+1\}$.

| | |
|---|---|
| $c(v_2) \leq c(v_0) \leq c(e_1) \leq c(v_1) \leq c(e_2)$ | $k \geq \max\{r+t+1, s+t+1, 3t+1\}$ |
| $c(v_0) \leq c(v_2) \leq c(e_1) \leq c(v_1) \leq c(e_2)$ | |
| $c(v_0) \leq c(e_1) \leq c(v_2) \leq c(v_1) \leq c(e_2)$ | |
| $c(v_0) \leq c(e_1) \leq c(v_1) \leq c(v_2) \leq c(e_2)$ | $k \geq \max\{r+3t+1, 2r+t+1, s+t+1\}$ |
| $c(v_0) \leq c(e_1) \leq c(v_1) \leq c(e_2) \leq c(v_2)$ | $k \geq \max\{2r+1, s+2t+1, 4t+1\}$ |

*2.2 If* $c(e_1) \leq c(v_0) \leq c(v_1) \leq c(e_2)$, *then* $k \geq \max\{r+2t+1, s+1\}$.

| | |
|---|---|
| $c(v_2) \leq c(e_1) \leq c(v_0) \leq c(v_1) \leq c(e_2)$ | $k \geq \max\{r+2t+1, s+1\}$ |
| $c(e_1) \leq c(v_2) \leq c(v_0) \leq c(v_1) \leq c(e_2)$ | |
| $c(e_1) \leq c(v_0) \leq c(v_2) \leq c(v_1) \leq c(e_2)$ | |
| $c(e_1) \leq c(v_0) \leq c(v_1) \leq c(v_2) \leq c(e_2)$ | $k \geq \max\{2r+2t+1, s+1\}$ |
| $c(e_1) \leq c(v_0) \leq c(v_1) \leq c(e_2) \leq c(v_2)$ | $k \geq \max\{r+3t+1, 2r+t+1, s+t+1\}$ |

*2.3 If* $c(e_1) \leq c(v_1) \leq c(v_0) \leq c(e_2)$, *then* $k \geq \max\{r+t+1, s+1, 2t+1\}$.

| | |
|---|---|
| $c(v_2) \leq c(e_1) \leq c(v_1) \leq c(v_0) \leq c(e_2)$ | $k \geq \max\{r+t+1, 2r+1, s+1, 2t+1\}$ |
| $c(e_1) \leq c(v_2) \leq c(v_1) \leq c(v_0) \leq c(e_2)$ | |
| $c(e_1) \leq c(v_1) \leq c(v_2) \leq c(v_0) \leq c(e_2)$ | $k \geq \max\{r+2t+1, s+1\}$ |
| $c(e_1) \leq c(v_1) \leq c(v_0) \leq c(v_2) \leq c(e_2)$ | |
| $c(e_1) \leq c(v_1) \leq c(v_0) \leq c(e_2) \leq c(v_2)$ | $k \geq \max\{r+2t+1, s+t+1, 3t+1\}$ |

*2.4 If* $c(e_1) \leq c(v_1) \leq c(e_2) \leq c(v_0)$, *then* $k \geq \max\{r+t+1, s+1, 2t+1\}$.

| | |
|---|---|
| $c(v_2) \leq c(e_1) \leq c(v_1) \leq c(e_2) \leq c(v_0)$ | $k \geq \max\{r+t+1, 2r+1, s+1, 2t+1\}$ |
| $c(e_1) \leq c(v_2) \leq c(v_1) \leq c(e_2) \leq c(v_0)$ | |
| $c(e_1) \leq c(v_1) \leq c(v_2) \leq c(e_2) \leq c(v_0)$ | $k \geq \max\{r+2t+1, s+1\}$ |
| $c(e_1) \leq c(v_1) \leq c(e_2) \leq c(v_2) \leq c(v_0)$ | $k \geq \max\{r+t+1, s+t+1, 3t+1\}$ |
| $c(e_1) \leq c(v_1) \leq c(e_2) \leq c(v_0) \leq c(v_2)$ | |

*Hence, for any path of order $n \geq 3$, by Lemma 2.22, the $[r,s,t]$-chromatic number is lower bounded by the minimum of all these values (where there is no need to consider values lower bounded by any other).*

$$\begin{aligned} k \geq \min\{ & \max\{r+2t+1, s+1\}, \\ & \max\{r+1, s+2t+1\}, \\ & \max\{r+t+1, s+t+1, 2t+1\}, \\ & \max\{r+t+1, 2r+1, s+1, 2t+1\}, \\ & r+s+t+1\}. \end{aligned}$$

*Observe that, if $r \leq s+2t$, $s \leq r+2t$ and $t \leq r+s$, then $k \geq \min\{r+2t+1, s+2t+1, \max\{r+t+1, s+t+1, 2t+1\}, \max\{r+t+1, 2r+1, s+1, 2t+1\}\}$.*

**Observation 3.3.** *If there are considered paths that have a chain of the form edge-vertex-edge-vertex-edge as a substructure (i.e. paths of order greater or equal to 4), then the same lower bound can be obtained, exchanging $r$ and $s$ in all cases (considering the same constellations with the following changes: $v_i \rightarrow e_{i+1}$ and $e_i \rightarrow v_i$).*

$$\begin{aligned} k \geq \min\{ & \max\{s+2t+1, r+1\}, \\ & \max\{s+1, r+2t+1\}, \\ & \max\{s+t+1, r+t+1, 2t+1\}, \\ & \max\{s+t+1, 2s+1, r+1, 2t+1\}, \\ & r+s+t+1\}. \end{aligned}$$

*If $r \leq s+2t$, $s \leq r+2t$ and $t \leq r+s$, then $k \geq \min\{s+2t+1, r+2t+1, \max\{s+t+1, r+t+1, 2t+1\}, \max\{s+t+1, 2s+1, r+1, 2t+1\}\}$.*

### 3.1.2 General Lower Bounds for $\mathcal{X}_{r,s,t}(P_n)$

For some concrete relations between the three constants $r$, $s$ and $t$, much better lower bounds than the general ones can be found (Lemma 2.25). They will be very useful to determine the $[r,s,t]$-chromatic number for paths and are presented in this subsection.

**Lemma 3.2.** *If $t < r \leq s < r+t$ and $2r \geq s+t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq s+t+1 \textit{ for all } n \geq 3.$$

*Proof.* By Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = s+t+1$. □

**Lemma 3.3.** *If $t \leq r \leq 2t$, $2r > s$ and $2r \leq s+t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq 2r+1 \textit{ for all } n \geq 3.$$

*Proof.* By Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = 2r+1$. □

**Lemma 3.4.** *If $s \geq r$, $s \geq t$ and $2r < 3t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq 2r + 1 \text{ for all } n \geq 4.$$

**Lemma 3.5.** *If $r \geq s$, $r \geq t$ and $2s < 3t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq 2s + 1 \text{ for all } n \geq 4.$$

To prove the previous lemmas it will be shown in detail a procedure that will be often used in this paper in similar situations. It will be called "symmetric replacement".

Observe that, in these lemmas, in the conditions and the consequences $r$ and $s$ have been exchanged in all cases. Introducing a new notation – for all $i$, let $x_i$,$y_i$ be the elements of the graph (vertices or edges) in the following order $...x_1y_1x_2y_2...$ and $N(x) = r$ if $x_i$ is a vertex for all $i$ and $N(x) = s$ if $x_i$ is an edge for all $i$ (analogously for $y$) – both can be enunciated together as follows.

Lemma 3.4/3.5:

If $N(x) \geq N(y)$, $N(x) \geq t$ and $2N(y) < 3t$, then

$$\mathcal{X}_{r,s,t}(P_n) \geq 2N(y) + 1 \text{ for all } n \geq 4.$$

Observe that in this case the minimum order of the path, for which the condition holds, is the same. But generally, this will not be the case, so the proof will be given and then, depending of the role of $x_i$ and $y_i$, this value will be fixed.

*Proof.* Suppose that $k \leq 2N(y)$. By Lemma 2.29, it may be assumed that $c(y_1), c(y_3) < c(y_2)$. Then $c(y_1), c(y_3) \leq N(y)$ and $N(y) + 1 \leq c(y_2) \leq 2N(y)$. By symmetry, assume that $c(x_2) < c(x_3)$. Then $c(y_2) < c(x_2) < c(x_3)$ is not possible because $N(x) + N(y) + t + 1 > 2N(y) + 1$.

Case 1, $c(x_2) < c(y_2) < c(x_3)$: Then $c(x_2) \leq 2N(y) - 2t$, $t+1 \leq c(y_2) \leq 2N(y) - t$ and $c(y_1), c(y_3) \leq N(y) - t$. Now $2N(y) - 3t < 0$ implies $c(y_1) > c(x_2)$ and $c(y_1) \geq t + 1$, a contradiction.

Case 2, $c(x_2) < c(x_3) < c(y_2)$: Then $c(x_2) \leq 2N(y) - t - N(x)$ and $N(x) + 1 \leq c(x_3) \leq 2N(y) - t$. Now $N(x) + t + 1 > N(y)$ implies $c(y_3) < c(x_3)$ and $c(y_3) \leq 2N(y) - 2t$. If there exists $x_4$, then $2N(y) - 3t < 0$ implies $c(x_4) > c(y_3)$ and $c(x_4) \geq t + 1$. Now $2N(y) - 2t < t + 1$ implies $c(x_4) > c(x_3)$ and $c(x_4) \geq 2N(x) + 1$, a contradiction.
If there exists no $x_4$, but there exists $x_1$ and $y_0$, then $2N(y) - 2t - N(x) \leq 2N(y) - 3t$ implies $c(y_1) > c(x_2)$ and $c(y_1) \geq t+1$. Now $2N(y) - t - 2N(x) \leq 2N(y) - 3t$ implies $c(x_1) > c(x_2)$ and $c(x_1) \geq N(x) + 1$. Next $N(y) - t < N(x) + 1$ implies $c(x_1) > c(y_1)$ and $c(x_1) \geq 2t + 1$. Then $c(y_0) < c(x_1)$ which implies $c(y_0) \leq 2N(y) - t$ and on the other hand $c(y_0) > c(y_1)$ implies $c(y_0) \geq N(y) + t + 1$, a contradiction to $N(y) \leq 2t$. ∎

Now both situations can be analyzed:
If $x_i$ is a vertex for all $i$ (Lemma 3.5), then $v_3$ ($x_4$) will always exist (because the existence of $e_3$ ($y_3$) was already assumed). Hence, the elements used are $e_1, v_1, e_2, v_2, e_3$ and $v_3$, in other words, a path of order 4. So $\mathcal{X}_{r,s,t}(P_n) \geq 2s+1$ for all $n \geq 4$.
On the other hand, if $x_i$ is an edge for all $i$ (Lemma 3.4), then $v_1, e_2, v_2, e_3, v_3$ and $e_4$ were used in one situation and $v_0, e_1, v_1, e_2, v_2, e_3$ and $v_3$ in the other. Hence, $\mathcal{X}_{r,s,t}(P_n) \geq 2r+1$ for all $n \geq 4$.

**Lemma 3.6.** *If $r \leq 2t$, $2r < 3t$ and $2r \leq 2t+s$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq 2r+1 \text{ for all } n \geq 5.$$

*Proof.* Observe that $2r \leq 2t+s$ implies $r \leq t+s/2 \leq t+s$.
Suppose $k \leq 2r$. By Lemma 2.29, it may be assumed that $c(v_0), c(v_2) < c(v_1)$. Hence, $c(v_0), c(v_2) \leq r$ and $r+1 \leq c(v_1) \leq 2r$. By symmetry, it can be supposed that $c(e_1) < c(e_2)$. Then $c(v_1) < c(e_1) < c(e_2)$ is not possible because $r+s+t+1 > 2r$.

Case 1, $c(e_1) < c(v_1) < c(e_2)$: Then $c(e_1) \leq 2r-2t$, $r+1 \leq c(v_1) \leq 2r-t$ and thus $c(v_0), c(v_2) \leq r-t$. Now $2r-3t < 0$ implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t+1$, a contradiction.

Case 2, $c(e_1) < c(e_2) < c(v_1)$: Then $c(e_1) \leq 2r-t-s$ and $s+1 \leq c(e_2) \leq 2r-t$. Now $2r-2t-s \leq 0$ implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t+1$. Hence $c(e_1) \leq r-t$.
If there exist $e_0$ and $v_{-1}$, then $r-t-s \leq 0$ implies $c(e_0) > c(e_1)$ and $c(e_0) \geq s+1$. Then, $r-t < s/2 < s+1$ implies $c(e_0) > c(v_0)$ and $c(e_0) \geq 2t+1$. Thus $c(v_{-1}) < c(e_0)$, which implies $c(v_{-1}) \leq 2r-t$. Furthermore $c(v_{-1}) > c(v_0)$, which implies $c(v_{-1}) \geq r+t+1$, a contradiction.
If there exist neither $e_0$ nor $v_{-1}$, but there exist $e_3$, $v_3$, $e_4$ and $v_4$, then $s+t+1 > r$ implies $c(v_2) < c(e_2)$ and $c(v_2) \leq 2r-2t$. Hence $c(e_2) \geq t+1$. Then $2r-3t < 0$ implies $c(e_3) > c(v_2)$ and $c(e_3) \geq t+1$. Now $2r-t-s \leq t$ implies $c(e_3) > c(e_2)$ and $c(e_3) \geq \max\{2s+1, s+t+1\}$. Thus, $s+2t+1 > 2r$ implies $c(v_3) < c(e_3)$ and $c(v_3) \leq 2r-t$. Furthermore $c(v_3) > c(v_2)$ implies $c(v_3) \geq r+1$. Then $c(e_3) \geq r+t+1$. Now $r+s+t+1 > 2r$ implies $c(e_4) < c(e_3)$ and $c(e_4) \leq 2r-s$. Next $r+t+1 > r+(r-s)+1 > 2r-s$ implies $c(e_4) < c(v_3)$ and $c(e_4) \leq 2r-2t$. Then $c(v_4) > c(e_4)$ implies $c(v_4) \geq t+1$ and $c(v_4) < c(v_3)$ implies $c(v_4) \leq 2r-2t$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(P_n) \geq 2r+1$ for all $n \geq 5$. □

**Lemma 3.7.** *If $2s < 3t$ and $2s \leq 2t+r$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq 2s+1 \text{ for all } n \geq 6.$$

*Proof.* It follows directly from the proof of Lemma 3.6 using "symmetric replacement". □

**Lemma 3.8.** *If $t \leq r \leq 2t$ and $s \leq r \leq s+t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq r+t+1 \text{ for all } n \geq 3.$$

*Proof.* By Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = r+t+1$. □

**Lemma 3.9.** *If $t \leq s \leq 2t$ and $r \leq s \leq r+t$, then*

$$\mathcal{X}_{r,s,t}(P_n) \geq s+t+1 \text{ for all } n \geq 4.$$

*Proof.* The lower bound given in Observation 3.3can be used and obtained $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = s+t+1$. □

## 3.2 $\mathcal{X}_{r,s,t}(P_n)$

In this section, the $[r,s,t]$-chromatic number of $P_n$ will be given for all possible triples $[r,s,t]$ and orders $n$.

**Theorem 3.10.** *If $P_2$ is a path of order 2, then*

$$\mathcal{X}_{r,s,t}(P_2) = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

*Proof.* Since just one edge is considered in this case, $\mathcal{X}_{r,s,t} = \mathcal{X}_{r,s',t}$ for all $s$ and $s'$. Then, the proof is a direct consequence of a Lemma given by Kemnitz and Marangio [11], that says that if $\mathcal{X}(G) = 2$ (which is the case), then

$$\mathcal{X}_{r,0,t}(G) = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

□

**Theorem 3.11.** *If $r \geq s+2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = r+1 \text{ for all } n \geq 3.$$

*Proof.* The following colouring

$$(..., t+1, \mathbf{r+1}, s+t+1, \mathbf{1}, t+1, \mathbf{r+1}, s+t+1, ...)$$

shows that $\mathcal{X}_{r,s,t}(P_n) \leq r+1$ for all $n$. So, by Corollary 3.1, it is concluded that $\mathcal{X}_{r,s,t}(P_n) = r+1$ for all $n \geq 2$. Observe that this includes $n = 2$ and $r \geq s+2t \geq 2t$, what was already proved. □

**Theorem 3.12.** *If $s \geq r + 2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = s + 1 \textit{ for all } n \geq 3.$$

*Proof.* Similarly to the proof of Theorem 3.11, the following colouring

$$(..., s+1, \mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, s+1, \mathbf{t{+}1}, 1, ...)$$

and Corollary 3.1 show that $\mathcal{X}_{r,s,t}(P_n) = s + 1$ for all $n \geq 3$. □

**Theorem 3.13.** *If $s \leq r < s + t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = r + t + 1 \textit{ for all } n \geq 3.$$

*Proof.* The following colouring

$$(..., t+1, \mathbf{1}, r+t+1, \mathbf{r{+}1}, t+1, \mathbf{1}, r+t+1, ...)$$

proves that $\mathcal{X}_{r,s,t}(P_n) \leq r+t+1$ for all $n$. Observation 3.2 implies that $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = r+t+1$ for $n \geq 3$. Hence, $\mathcal{X}_{r,s,t}(P_n) = r+t+1$ for all $n \geq 3$. □

**Theorem 3.14.** *If $s + t \leq r < s + 2t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = s + 2t + 1 \textit{ for all } n \geq 3.$$

*Proof.* The following colouring

$$(..., t+1, \mathbf{1}, s+t+1, \mathbf{s{+}2t{+}1}, t+1, \mathbf{1}, s+t+1, ...)$$

shows that $\mathcal{X}_{r,s,t}(P_n) \leq s + 2t + 1$ for all $n$, and a lower bound for $n \geq 3$ is given by Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = s+2t+1$. Hence, $\mathcal{X}_{r,s,t}(P_n) = s + 2t + 1$ for all $n \geq 3$. □

**Theorem 3.15.** *If $r \leq s < r + t$ and $s \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} 2r+1 & \textit{if } \ 2r < s+t \textit{ for } n = 3; \\ s+t+1 & \textit{otherwise.} \end{cases}$$

*Proof.* Observation 3.3 shows that $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = s+t+1$ for $n \geq 4$ and an upper bound is given by the following colouring

$$(..., s+1, \mathbf{t{+}1}, 1, \mathbf{s{+}t{+}1}, s+1, \mathbf{t{+}1}, 1, ...).$$

Hence, $\mathcal{X}_{r,s,t}(P_n) = s + t + 1$ for all $n \geq 4$.

For $n = 3$, if $2r \geq s + t$, Lemma 3.2 proves that $\mathcal{X}_{r,s,t}(P_3) = s + t + 1$. And if $2r < s + t$, the following colouring

$$(\mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1})$$

(observe that $r+t > s \geq 2t$ implies $r > t$, therefore $2r > s$) demonstrates that $\mathcal{X}_{r,s,t}(P_3) \leq 2r+1$. Then $r < 2t$ ($2r < s+t$ implies $r < s-r+t < 2t$ because $s < r+t$, hence $s-r < t$) and $s < 2r$ ($s < r+t < 2r$). Hence, Lemma 3.3 can be applied and it shows that $\mathcal{X}_{r,s,t}(P_3) = 2r+1$. □

**Theorem 3.16.** *If $r+t \leq s < r+2t$ and $s \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} s+1 & \text{if } \ r < 2t \text{ and } (r < t \text{ or } 2r \leq s) \text{ for } n = 3; \\ 2r+1 & \text{if } \ s < 2r < 4t \text{ and } r \geq t \text{ for } n = 3; \\ r+2t+1 & \text{otherwise.} \end{cases}$$

*Proof.* Observation 3.3 gives the lower bound $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = r+2t+1$ and then the following colouring

$$(..., r+2t+1, \mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, r+2t+1, \mathbf{t{+}1}, 1, ...)$$

implies that $\mathcal{X}_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 4$.
For $n = 3$, if $r \geq 2t$, Observation 3.2 shows that $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = r+2t+1$. Hence $\mathcal{X}_{r,s,t}(P_3) = r+2t+1$ with the previous colouring. If $r < 2t$, then one possible colouring would be

$$(\mathbf{1}, s+1, \max\{\mathbf{t+1}, \mathbf{r+1}\}, 1, \max\{\mathbf{2r+1}, \mathbf{r+t+1}\}).$$

Hence, if $t > r$ or ($r \geq t$ and $2r \leq s$), by Corollary 3.1 $\mathcal{X}_{r,s,t}(P_3) = s+1$. On the other hand, if $r \geq t$ and $2r > s$, by the colouring and Lemma 3.3 (because $2r < (r+t)+t \leq s+t$), $\mathcal{X}_{r,s,t}(P_3) = 2r+1$. □

**Theorem 3.17.** *For $t < r, s < 2t$,*

1. *If $n = 3$, then*

$$\mathcal{X}_{r,s,t}(P_3) = \begin{cases} r+t+1 & \text{if } \ s < r; \\ s+t+1 & \text{if } \ r \leq s \text{ and } 2r > s+t; \\ 2r+1 & \text{if } \ r \leq s \text{ and } 2r \leq s+t. \end{cases}$$

2. *If ($3t \leq 2r$, $3t \leq 2s$ for $n \geq 4$) or ($3t \leq 2r$, $3t > 2s$ for $n \geq 5$) or ($3t > 2r$, $3t \leq 2s$ for $n \geq 6$), then*

$$\mathcal{X}_{r,s,t}(P_n) = 3t+1.$$

3. *If $3t \leq 2r$ and $3t > 2s$ for $n = 4$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} 2s+1 & \text{if } \ 2s > r+t; \\ r+t+1 & \text{if } \ 2s \leq r+t. \end{cases}$$

*4. If $3t > 2r$ and $3t \le 2s$ for $n = 4$ or $5$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} 2r+1 & \text{if } 2r > s+t; \\ s+t+1 & \text{if } 2r \le s+t. \end{cases}$$

*5. If $3t > 2r$ and $3t > 2s$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} 2r+1 & \text{if } (s < r \text{ for } n \ge 5) \text{ or } (r < s \text{ and } 2r > s+t \text{ for } n = 4 \text{ or } 5); \\ 2s+1 & \text{if } (r < s \text{ for } n \ge 6) \text{ or } (s < r \text{ and } 2s > r+t \text{ for } n = 4); \\ r+t+1 & \text{if } s < r \text{ and } 2s \le r+t \text{ for } n = 4; \\ s+t+1 & \text{if } r < s \text{ and } 2r \le s+t \text{ for } n = 4 \text{ or } 5. \end{cases}$$

*Proof.* Observe that in this case, there is a triple $[r, s, t]$ which satisfies the conditions for Lemmas 3.8 and 3.9, hence the following lower bounds hold:
$\mathcal{X}_{r,s,t}(P_n) \ge r+t+1$ for all $n \ge 3$, if $s \le r$, and $\mathcal{X}_{r,s,t}(P_n) \ge \max\{r+t+1, s+t+1\}$ for all $n \ge 4$.

(1) If $\mathbf{n = 3}$, then

If $s < r$, then the following colouring

$$(\mathbf{r{+}1}, r+t+1, \mathbf{1}, t+1, \mathbf{2t{+}1})$$

and the lower bound show that $\mathcal{X}_{r,s,t}(P_3) = r+t+1$.

If $r \le s$, then the following colouring

$$(\mathbf{s{+}1}, s+t+1, \mathbf{1}, t+1, \mathbf{2t{+}1})$$

proves that $\mathcal{X}_{r,s,t}(P_3) \le s+t+1$. If $2r \ge s+t$, then by Lemma 3.2, $\mathcal{X}_{r,s,t}(P_3) = s+t+1$. And if $2r < s+t$, then $P_3$ could be coloured like

$$(\mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}),$$

which together with Lemma 3.3 shows that $\mathcal{X}_{r,s,t}(P_3) = 2r+1$.

(2) If $\mathbf{3t \le 2r}$ **or** $\mathbf{3t \le 2s}$, then the colouring

$$(..., 2t+1, \mathbf{t{+}1}, 1, \mathbf{3t{+}1}, 2t+1, \mathbf{t{+}1}, 1, ...)$$

implies that $\mathcal{X}_{r,s,t}(P_n) \le 3t+1$ for all $n$.

Suppose $k \le 3t$. Then, if $3t \le 2r$, by Lemma 2.29, it may be supposed that $c(v_0), c(v_2) < c(v_1)$. Hence, $c(v_0), c(v_2) \le 3t - r$ and $r+1 \le c(v_1) \le 3t$. By symmetry, assume that $c(e_1) < c(e_2)$. Then $c(e_2) > c(e_1) > c(v_1)$ is not possible because $r+s+t+1 > 3t$.

Case 1, $c(e_1) < c(v_1) < c(e_2)$: Then $c(e_1) \le t$, $r+1 \le c(v_1) \le 2t$ and thus $c(v_0), c(v_2) \le 2t - r$. Then $c(v_0) > c(e_1)$, which implies $c(v_0) \ge t+1$, a contradiction.

Case 2, $c(e_1) < c(e_2) < c(v_1)$: Then $c(e_1) \leq 2t - s$ and $s + 1 \leq c(e_2) \leq 2t$. This implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$.
If there exist $e_0$ and $v_{-1}$, then $c(e_0) > c(e_1)$ and $c(e_0) \geq s + 1$, thus $c(e_0) > c(v_0)$ and $c(e_0) \geq 2t + 1$. Therefore, $c(v_{-1}) < c(e_0)$ and $c(v_{-1}) \leq 2t$, hence $c(v_{-1}) < c(v_0)$ and $c(v_{-1}) \leq 3t - 2r \leq 0$, which is a contradiction.
If there exist neither $e_0$ nor $v_{-1}$, but there exist $e_3$, $v_3$, $e_4$ and $v_4$, then $c(v_2) < c(e_2)$ and $c(v_2) \leq t$. Therefore, $c(e_3) > c(v_2)$ and $c(e_3) \geq t+1$, which implies $c(e_3) > c(e_2)$ and $c(e_3) \geq 2s + 1$. And this holds if and only if $3t > 2s$.
In this case, $c(v_3) < c(e_3)$ implies $c(v_3) \leq 2t$ and $c(v_3) > c(v_2)$ implies $c(v_3) \geq r + 1$. Hence, $c(e_3) \geq r + t + 1$. Then $c(e_4) < c(e_3)$ and $c(e_4) \leq 3t - s$ imply $c(e_4) < c(v_3)$ and $c(e_4) \leq t$. Therefore, $c(v_4) > c(e_4)$ and $c(v_4) \geq t + 1$ which imply $c(v_4) > c(v_3)$ and $c(v_4) \geq 2r + 1 \geq 3t$, a contradiction.

Hence, if $3t \leq 2r$ and $3t \leq 2s$, then $\mathcal{X}_{r,s,t}(P_n) = 3t + 1$ for all $n \geq 4$, and if $3t \leq 2r$ and $3t > 2s$, then $\mathcal{X}_{r,s,t}(P_n) = 3t + 1$ for all $n \geq 5$.

If $3t \leq 2s$, then the application of "symmetric replacement" in the proof of the case $3t \leq 2r$ demonstrates that, if $3t \leq 2s$ and $3t > 2r$, then $\mathcal{X}_{r,s,t}(P_n) = 3t + 1$ for all $n \geq 6$.

(3) If $\mathbf{3t} \leq \mathbf{2r}$ **and** $\mathbf{3t} > \mathbf{2s}$ **for** $\mathbf{n} = \mathbf{4}$, then the colouring

$$(\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, s + 1, \mathbf{1}, \max\{2s + 1, r + t + 1\}, \mathbf{r{+}1})$$

implies that $\mathcal{X}_{r,s,t}(P_4) = 2s + 1$, if $2s > r + t$ by Lemma 3.5. And if $2s \leq r + t$, then by the lower bound, $\mathcal{X}_{r,s,t}(P_4) = r + t + 1$.

(4) If $\mathbf{3t} > \mathbf{2r}$ **and** $\mathbf{3t} \leq \mathbf{2s}$ **for** $\mathbf{n} = \mathbf{4}$ **or** $\mathbf{5}$, then the colouring

$$(\mathbf{s{+}t{+}1}, t + 1, \mathbf{1}, s + t + 1, \mathbf{r{+}1}, 1, \max\{\mathbf{2r + 1}, \mathbf{s + t + 1}\}, s + 1, \mathbf{1})$$

shows that $\mathcal{X}_{r,s,t}(P_n) = 2r + 1$ for $n = 4$ or 5, if $2r > s + t$ by Lemma 3.4. And if $2r \leq s + t$, then by the lower bound, $\mathcal{X}_{r,s,t}(P_n) = s + t + 1$ for $n = 4$ or 5.

(5) If $\mathbf{3t} > \mathbf{2r}$ **and** $\mathbf{3t} > \mathbf{2s}$, then

If $s < r$, then the colouring

$$(..., 2r + 1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}, r + 1, \mathbf{1}, 2r + 1, ...)$$

and Lemma 3.6 show that $\mathcal{X}_{r,s,t}(P_n) = 2r + 1$ for all $n \geq 5$. If $n = 4$, then it is basically the same situation as in case *c)*. Hence, if $2s > r+t$, then $\mathcal{X}_{r,s,t}(P_4) = 2s+1$ and, if $2s \leq r + t$, then $\mathcal{X}_{r,s,t}(P_4) = r + t + 1$.

If $r < s$, then by Lemma 3.7, $\mathcal{X}_{r,s,t}(P_n) = 2s + 1$ for all $n \geq 6$ with the colouring

$$(..., 2s + 1, \mathbf{s{+}1}, 1, \mathbf{2s{+}1}, s + 1, \mathbf{1}, 2s + 1, ...)$$

and, if $n = 4$ or 5, then the same proof as in case (4) could be used. Hence, if $2r > s + t$, then $\mathcal{X}_{r,s,t}(P_n) = 2r + 1$ for $n = 4$ or 5, and if $2r \leq s + t$, then $\mathcal{X}_{r,s,t}(P_n) = s + t + 1$ for $n = 4$ or 5. $\square$

**Theorem 3.18.** *If $s \le t \le r < 2t$ and $s < r$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} s+2t+1 & \text{if } s \le 2r-2t \text{ and } ((r \ge s+t \text{ and } n \ge 3) \text{ or } n \ge 5); \\ 2r+1 & \text{if } s > 2r-2t \text{ and } n \ge 5; \\ r+t+1 & \text{otherwise.} \end{cases}$$

*Proof.* *a)* If $\mathbf{s \le 2r - 2t}$, then the colouring

$$(..., s+t+1, \mathbf{1}, t+1, \mathbf{s{+}2t{+}1}, s+t+1, \mathbf{1}, t+1, ...)$$

proves that $\mathcal{X}_{r,s,t}(P_n) \le s+2t+1$ for all $n$.

If $r \ge s+t$, Observation 3.2 gives the lower bound $k \ge \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = s+2t+1$, for $n \ge 3$. Hence, $\mathcal{X}_{r,s,t}(P_n) = s+2t+1$ for all $n \ge 3$.

If $r < s+t$, suppose $k \le s+2t$, which is at most 2r. Then, by Lemma 2.29, it may be assumed that $c(v_0), c(v_2) < c(v_1)$. Hence $c(v_0), c(v_2) \le s+2t-r$ (observe that $s+2t-r > t$) and $r+1 \le c(v_1) \le s+2t$. By symmetry, assume that $c(e_1) < c(e_2)$, then $c(e_1) \le 2t$ and $s+1 \le c(e_2) \le s+2t$.

Case 1, $c(e_2) > c(v_1)$: Then $r+1 \le c(v_1) \le s+t$ and thus $c(e_1) < c(v_1)$ and $c(e_1) \le s$. Then $c(v_0) > c(e_1)$ implies $c(v_0) \ge t+1$ and now $c(v_0) < c(v_1)$ implies $c(v_0) \le s+t-r$, a contradiction.

Case 2, $c(e_2) < c(v_1)$: Then $s+1 \le c(e_2) \le s+t$ and $c(e_1) \le t$. Which implies $c(v_0) > c(e_1)$, $t+1 \le c(v_0) \le s+2t-r$ and $e_1 \le s+t-r$.
If there exist $e_0$ and $v_{-1}$, then $c(e_0) > c(e_1)$ and $c(e_0) \ge s+1$. Then $c(e_0) > c(v_0)$ and $c(e_0) \ge 2t+1$. And thus $c(v_{-1}) < c(e_0)$ which implies $c(v_{-1}) \le s+t$ and $c(v_{-1}) > c(v_0)$ that implies $c(v_{-1}) \ge r+t+1$, a contradiction.
If there exist neither $e_0$ nor $v_{-1}$, but there exist $e_3$, $v_3$, $e_4$ and $v_4$, then $c(v_2) < c(e_2)$ implies $c(v_2) \le s$ and $c(e_2) \ge t+1$. Then $c(e_3) > c(v_2)$ and $c(e_3) \ge t+1$ and thus, $c(e_3) > c(e_2)$ and $c(e_3) \ge s+t+1$. Then $c(v_3) < c(e_3)$, hence $c(v_3) \le s+t$. And $c(v_3) > c(v_2)$ thus $c(v_3) \ge r+1$, which implies $c(e_3) \ge r+t+1$. Then $c(e_4) < c(e_3)$ and $c(e_4) \le 2t$. Hence $c(e_4) < c(v_3)$, $c(e_4) \le s$ and $c(v_4) > c(e_4)$, which implies $c(v_4) \ge t+1$, $c(v_4) > c(v_3)$ and $c(v_4) \ge 2r+1$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(P_n) = s+2t+1$ for all $n \ge 5$.

*b)* If $\mathbf{s > 2r - 2t}$, then the colouring

$$(..., r+1, \mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}, r+1, ...)$$

implies that $\mathcal{X}_{r,s,t}(P_n) \le 2s+1$ for all $n$. And Lemma 3.6 can be applied, hence $\mathcal{X}_{r,s,t}(P_n) = 2r+1$ for all $n \ge 5$.

*c)* If $\mathbf{s > 2r - 2t}$ **or** $(\mathbf{s \le 2r - 2t}$ **and** $\mathbf{r < s + t})$ **and** $\mathbf{n \le 4}$, then the colouring

$$(\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, t+1, \mathbf{1}, r+t+1, \mathbf{r{+}1})$$

and Lemma 3.8 show that $\mathcal{X}_{r,s,t}(P_n) = r+t+1$ for $n = 3, 4$. □

**Theorem 3.19.** *If $r \leq t \leq s < 2t$ and $r < s$, then*

$$\mathcal{X}_{r,s,t}(P_n) = \begin{cases} r+2t+1 & \text{if } r \leq 2s-2t \text{ and } ((s \geq r+t \text{ and } n \geq 4) \text{ or } n \geq 6); \\ 2s+1 & \text{if } r > 2s-2t \text{ and } n \geq 6; \\ 2t+1 & \text{if } n = 3; \\ s+t+1 & \text{otherwise.} \end{cases}$$

*Proof.* *a)* If $\mathbf{r \leq 2s - 2t}$, then the colouring

$$(..., r+2t+1, \mathbf{t+1}, 1, \mathbf{r+t+1}, r+2t+1, \mathbf{t+1}, 1, ...)$$

shows that $\mathcal{X}_{r,s,t}(P_n) \leq r+2t+1$ for all $n$.

If $r \geq r+t$, by Observation 3.3, $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = r+2t+1$. Hence, $\mathcal{X}_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 4$.

If $s < r+t$, then applying "symmetric replacement" in the proof of Theorem 3.18 for $r < s+t$, it follows that $\mathcal{X}_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 6$.

*b)* If $\mathbf{r > 2s - 2t}$, then the colouring

$$(..., s+1, \mathbf{1}, 2s+1, \mathbf{s+1}, 1, \mathbf{2s+1}, s+1, ...)$$

demonstrates that $\mathcal{X}_{r,s,t}(P_n) \leq 2s+1$ for all $n$. Hence, by Lemma 3.7 $\mathcal{X}_{r,s,t}(P_n) = 2s+1$ for all $n \geq 6$.

*c)* If $\mathbf{r > 2s - 2t}$ **or** $(\mathbf{r \leq 2s - 2t}$ **and** $\mathbf{s < r + t})$ **and** $\mathbf{n = 4}$ **or** $\mathbf{5}$, then the colouring

$$(\mathbf{2t+1}, t+1, \mathbf{1}, s+t+1, \mathbf{s+1}, 1, \mathbf{s+t+1}, s+1, \mathbf{1})$$

and Lemma 3.9 show that $\mathcal{X}_{r,s,t}(P_n) = s+t+1$ for $n = 4$ or 5.

*d)* If $\mathbf{n = 3}$, then the colouring

$$(\mathbf{1}, 2t+1, \mathbf{t+1}, 1, \mathbf{2t+1})$$

implies that $\mathcal{X}_{r,s,t}(P_3) \leq 2t+1$. By Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2t+1\} = 2t+1$. Hence, $\mathcal{X}_{r,s,t}(P_3) = 2t+1$. □

**Theorem 3.20.** *If $r, s \leq t < r+s$, then*

$$\mathcal{X}_{r,s,t}(P_n) = 2t+1 \text{ for all } n \geq 3.$$

*Proof.* The colouring

$$(..., t+1, \mathbf{1}, 2t+1, \mathbf{t+1}, 1, \mathbf{2t+1}, t+1, ...)$$

shows that $\mathcal{X}_{r,s,t}(P_n) \leq 2t+1$ for all $n$. By Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, 2t+1, 2t+1\} = 2t+1$ for all $n \geq 3$. Hence, $\mathcal{X}_{r,s,t}(P_n) = 2t+1$ for all $n \geq 3$. □

**Theorem 3.21.** *If $t \geq r+s$, then*

$$\mathcal{X}_{r,s,t}(P_n) = r+s+t+1 \text{ for all } n \geq 3.$$

*Proof.* By Corollary 3.1, the bound $\mathcal{X}_{r,s,t}(P_n) \leq r+s+t+1$ holds for all $n$, because the colouring

$$(..., r+s+t+1, \mathbf{1}, r+t+1, \mathbf{r+1}, r+s+t+1, \mathbf{1}, r+t+1, ...)$$

is always possible. Then, by Observation 3.2, $k \geq \min\{r+2t+1, s+2t+1, 2t+1, 2t+1, r+s+t+1\} = r+s+t+1$. Hence, $\mathcal{X}_{r,s,t}(P_n) = r+s+t+1$ for all $n \geq 3$. □

All results presented in this chapter are summarized in Table 3.1.

| Conditions | | | Order | $\mathcal{X}_{r,s,t}(P_n)$ |
|---|---|---|---|---|
| $r \geq 2t$ | | | 2 | r+1 |
| $t \leq r < 2t$ | | | 2 | 2t+1 |
| $r < t$ | | | 2 | r+t+1 |
| $r \geq s + 2t$ | | | ≥3 | r+1 |
| $s \geq r + 2t$ | | | ≥3 | s+1 |
| $(s \leq r < s + t) \wedge (r \geq 2t)$ | | | ≥3 | r+t+1 |
| $(s + t \leq r < s + 2t) \wedge (r \geq 2t)$ | | | ≥3 | s+2t+1 |
| $(r \leq s < r + t) \wedge (s \geq 2t)$ | | $2r < s + t$ | 3 | 2r+1 |
| | | $2r \geq s + t \vee n \geq 4$ | ≥3 | s+t+1 |
| $(r + t \leq s < r + 2t) \wedge (s \geq 2t)$ | | $r < 2t \wedge (r < t \vee 2r \leq s)$ | 3 | s+1 |
| | | $s < 2r < 4t \wedge r \geq t$ | 3 | 2r+1 |
| | | $r \geq 2t \vee n \geq 4$ | ≥3 | r+2t+1 |
| $t < r, s < 2t$ | $s < r$ | | 3 | r+t+1 |
| | $r \leq s$ | $2r > s + t$ | | s+t+1 |
| | | $2r \leq s + t$ | | 2r+1 |
| | $(3t \leq 2r \wedge 3t \leq 2s)\vee$ $(3t \leq 2r \wedge 3t > 2s \wedge n \geq 5)\vee$ $(3t > 2r \wedge 3t \leq 2s \wedge n \geq 6)$ | | ≥4 | 3t+1 |
| | $3t \leq 2r \wedge 3t > 2s$ | $2s > r + t$ | 4 | 2s+1 |
| | | $2s \leq r + t$ | | r+t+1 |
| | $3t > 2r \wedge 3t \leq 2s$ | $2r > s + t$ | 4,5 | 2r+1 |
| | | $2r \leq s + t$ | | s+t+1 |
| | $3t > 2r \wedge 3t > 2s$ | $(r < s \wedge 2r > s + t \wedge n = 4, 5)$ $\vee(s < r \wedge n \geq 5)$ | ≥4 | 2r+1 |
| | | $(s < r \wedge 2s > r + t \wedge n = 4)$ $\vee(r < s \wedge n \geq 6)$ | ≥4 | 2s+1 |
| | | $s < r \wedge 2s \leq r + t$ | 4 | r+t+1 |
| | | $r < s \wedge 2r \leq s + t$ | 4,5 | s+t+1 |
| $(s \leq t \leq r < 2t)$ $\wedge(s < r)$ | $r < s + t \vee s > 2r - 2t$ | | 3,4 | r+t+1 |
| | $s \leq 2r - 2t \wedge (r \geq s + t \vee n \geq 5)$ | | ≥3 | s+2t+1 |
| | $s > 2r - 2t$ | | ≥5 | 2r+1 |
| $(r \leq t \leq s < 2t)$ $\wedge(r < s)$ | | | 3 | 2t+1 |
| | $s < r + t \vee r > 2s - 2t$ | | 4,5 | s+t+1 |
| | $r \leq 2s - 2t \wedge (s \geq r + t \vee n \geq 6)$ | | ≥4 | r+2t+1 |
| | $r > 2s - 2t$ | | ≥6 | 2s+1 |
| $r, s \leq t < r + s$ | | | ≥3 | 2t+1 |
| $t \geq r + s$ | | | ≥3 | r+s+t+1 |

Table 3.1: $[r, s, t]$-chromatic number of paths

# Chapter 4

# $[r, s, t]$-Colouring of Cycles

Once the values of the $[r, s, t]$-chromatic number of paths for all possible constellations $[r, s, t]$ were known, a logical next step was to find them for cycles, because, as noted in Observation 4.1, the values for paths can be used as lower bounds (often sharp) for them.

**Observation 4.1.** *Because of Lemma 2.22, the values of $\mathcal{X}_{r,s,t}(P_n)$ given in the previous chapter can be considered as lower bounds of the corresponding $\mathcal{X}_{r,s,t}(C_n)$, where $C_n$ is the cycles with $n$ vertices.*

As shown in Lemma 2.18, the total-chromatic number of a cycle, $C_n$, depends on the value of its order modulo 3. This is, if $n$ is zero modulo 3, $\mathcal{X}_T(C_n) = 3$, otherwise $\mathcal{X}_T(C_n) = 4$. This fact implies that there is a monotone sequence of colours of three (respectively four) elements of the cycle, where each element is in contact with those elements whose colours are the immediately precedent and the immediately following of its colour in the sequence. In case $n \neq 0(mod\ 3)$, this property will be very useful in order to find the $[r, s, t]$-chromatic number and it is presented in the following Corollary.

**Corollary 4.1.** *From Lemma 2.18 it follows that if $n \neq 0(mod\ 3)$, then there is a subpath of the cycle with one of the following constellations, which because of Lemma 2.22 give lower bounds for $k := \mathcal{X}_{r,s,t}(C_n)$ (reduction made by application of Lemma 2.29):*

**1.** *If $c(v_0) \leq c(v_1) \leq c(v_2) \leq c(v_3)$, then $k \geq 3r + 1$.*
**2.** *If $c(v_0) \leq c(v_1) \leq c(v_2) \leq c(e_3)$ (**2(1)**) or*
   *$c(v_0) \leq c(v_1) \leq c(v_2) \leq c(e_2)$ (**2(2)**), then $k \geq 2r + t + 1$.*
**3.** *If $c(v_0) \leq c(v_1) \leq c(e_2) \leq c(v_2)$, then $k \geq \max\{r + 2t + 1, 2r + 1\}$.*
**4.** *If $c(v_0) \leq c(v_1) \leq c(e_2) \leq c(e_3)$, then $k \geq r + s + t + 1$.*
**5.** *If $c(v_0) \leq c(e_1) \leq c(v_1) \leq c(e_2)$, then $k \geq \max\{3t + 1, r + t + 1, s + t + 1\}$.*
**6.** *If $c(e_1) \leq c(v_0) \leq c(v_1) \leq c(e_2)$, then $k \geq \max\{r + 2t + 1, s + 1\}$.*
**7.** *If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{s + 2t + 1, r + 1\}$.*

**8.** *If* $c(e_1) \leq c(v_1) \leq c(e_2) \leq c(e_3)$*, then* $k \geq \max\{s+2t+1, 2s+1\}$.
**9.** *If* $c(v_0)) \leq c(e_1) \leq c(e_2) \leq c(e_3)$ *(***9(1)***) or*
$c(v_1)) \leq c(e_1) \leq c(e_2) \leq c(e_3)$ *(***9(2)***), then* $k \geq 2s+t+1$.
**10.** *If* $c(e_1) \leq c(e_2) \leq c(e_3) \leq c(e_4)$*, then* $k \geq 3s+1$.

**Notation 2.** *For cycles the same notation as for paths given in Notation 1 will be used, adding some extra symbology in order to be able to express the special structure of the* $[r, s, t]$*-colouring of a cycle. For instance, the chain of colours that is underlined is the substructure that should be repeated as often as needed in order to cover the cycle. In some cases, there will also be another chain of colours that will be double underlined and needs to be placed just once.*

As noted in Section 2.1, cycles have different chromatic number, edge-chromatic number and total-chromatic number, depending on the parity of their order. Due to this fact, cycles with even order and those with odd order will be treated separately.

## 4.1 Cycles of even order

In this section only cycles with an even number of vertices will be considered.

Since cycles of even order, $C_{2n}$, are bipartite graphs, $\mathcal{X}(C_{2n}) = 2$ and $\mathcal{X}'(C_{2n}) = \Delta(C_{2n}) = 2$, as observed in Lemma 2.6 and Theorem 2.11. With these values, Lemma 2.25 yields to the following bounds.

**Corollary 4.2.** *For any cycle of even order,* $C_{2n}$*, it holds that*

$$\max\{r+1, s+1, t+2 \leq \mathcal{X}_{r,s,t}(C_{2n}) \leq r+s+t+1$$

*for all* $n \geq 2$.

Then all possible constellations of $[r, s, t]$ will be studied.

**Theorem 4.3.** *If* $r \geq s+2t$*, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = r+1 \textit{ for all } n \geq 2.$$

*Proof.* By Corollary 4.2, $\mathcal{X}_{r,s,t}(C_{2n}) \geq r+1$ for all $n \geq 2$ and the following colouring

$$(..., \underline{\mathbf{1}, t+1, \mathbf{r+1}, s+t+1}, \mathbf{1}, ...)$$

verifies that $\mathcal{X}_{r,s,t}(C_{2n}) = r+1$ for all $n \geq 2$. □

**Theorem 4.4.** *If* $s \geq r+2t$*, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = s+1 \textit{ for all } n \geq 2.$$

*Proof.* Like in the previous proof, from Corollary 4.2 follows the lower bound $\mathcal{X}_{r,s,t}(C_{2n}) \geq s+1$ for all $n \geq 2$. The upper bound is given by the colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, s+1}, \mathbf{t{+}1}, ...).$$

Hence, $\mathcal{X}_{r,s,t}(C_{2n}) = s+1$ for all $n \geq 2$. ☐

**Theorem 4.5.** *If $s \leq r < s+t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = r+t+1 \textit{ for all } n \geq 2.$$

*Proof.* The following colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, r+1}, \mathbf{t{+}1}, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n}) \leq r+t+1$ for all $n$ and by Observation 4.1, $\mathcal{X}_{r,s,t}(C_{2n}) \geq r+t+1$ for all $2n \geq 3$. So, $\mathcal{X}_{r,s,t}(C_{2n}) = r+t+1$ for all $n \geq 2$. ☐

**Theorem 4.6.** *If $s+t \leq r < s+2t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = s+2t+1 \textit{ for all } n \geq 2.$$

*Proof.* Observation 4.1 gives the lower bound $\mathcal{X}_{r,s,t}(C_{2n}) \geq s+2t+1$ for all $2n \geq 3$. Then the following colouring

$$(..., \underline{\mathbf{1}, s+t+1, \mathbf{s{+}2t{+}1}, t+1}, \mathbf{1}, ...)$$

shows that $\mathcal{X}_{r,s,t}(C_{2n}) = s+2t+1$ for all $n \geq 2$. ☐

**Theorem 4.7.** *If $r \leq s < r+t$ and $s \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = s+t+1 \textit{ for all } n \geq 2.$$

*Proof.* By Observation 4.1 (observe that paths of length at least 4 are subgraphs of the cycle), $\mathcal{X}_{r,s,t}(C_{2n}) \geq s+t+1$ for all $n \geq 2$. And from the following colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{s{+}t{+}1}, s+1}, \mathbf{t{+}1}, ...)$$

it follows that $\mathcal{X}_{r,s,t}(C_{2n}) = s+t+1$ for all $n \geq 2$. ☐

**Theorem 4.8.** *If $r+t \leq s < r+2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = r+2t+1 \textit{ for all } n \geq 2.$$

*Proof.* Like in the previous proofs, Observation 4.1 implies that $\mathcal{X}_{r,s,t}(C_{2n}) \geq r+2t+1$ for all $n \geq 2$. The upper bound is given by the colouring

$$(..., \underline{\mathbf{t{+}1}, r+2t+1, \mathbf{r{+}t{+}1}, 1}, \mathbf{t{+}1}, ...).$$

Hence, $\mathcal{X}_{r,s,t}(C_{2n}) = r+2t+1,$ for all $n \geq 2$. ☐

**Theorem 4.9.** *For $t < r, s < 2t$,*

1. *If $3t \leq 2r$ or $3t \leq 2s$, then*
$$\mathcal{X}_{r,s,t}(C_{2n}) = 3t + 1 \text{ for all } n \geq 2.$$

2. *If $3t > 2r$ and $3t > 2s$, then*
$$\mathcal{X}_{r,s,t}(C_{2n}) = \begin{cases} 2r+1 & \text{if} \quad s < r \text{ and } 2n = 3k; \\ 2s+1 & \text{if} \quad r \leq s \text{ and } 2n = 3k; \\ 3t+1 & \text{if} \quad 2n \neq 3k. \end{cases}$$

*Proof.* (1) If (**3t** $\leq$ **2r and 3t** $\leq$ **2s for 2n** $\geq$ **4**) **or** (**3t** $\leq$ **2r and 3t** $>$ **2s for 2n** $\geq$ **5**) **or** (**3t** $>$ **2r and 3t** $\leq$ **2s for 2n** $\geq$ **6**), then from Observation 4.1 it follows that $\mathcal{X}_{r,s,t}(C_{2n}) \geq 3t + 1$. Then the colouring

$$(..., \underline{\mathbf{t{+}1}, 2t+1, \mathbf{3t{+}1}, 1}, \mathbf{t{+}1}, ...).$$

proves that $\mathcal{X}_{r,s,t}(C_{2n}) = 3t + 1$ for the mentioned cases.

If $3t \leq 2r$ and $3t > 2s$ for $2n = 4$, suppose $k \leq 3t \leq 2r$. Then by Lemma 2.29, it may be supposed that $c(v_0), c(v_2) < c(v_1)$. Hence $c(v_0), c(v_2) \leq 3t - r$ and $r + 1 \leq c(v_1) \leq 3t$. By symmetry, assume that $c(e_1) < c(e_2)$. Then it is not possible $c(e_2) > c(e_1) > c(v_1)$, because $3t < r + s + t + 1$.

Case 1, $c(e_1) < c(v_1) < c(e_2)$: Then $c(e_1) \leq t$, $c(v_1) \leq 2t$ and $c(v_0), c(v_2) \leq 2t - r$. Hence $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$, a contradiction.

Case 2, $c(e_1) < c(e_2) < c(v_1)$: Then $c(e_1) \leq 2t - s$, which implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$. Now $c(e_4) > c(e_1)$ implies $c(e_4) \geq s + 1$, $c(e_4) > c(v_0)$ and $c(e_4) \geq 2t + 1$. Hence $c(v_3) < c(e_4)$ which implies $c(v_3) \leq 2t$, $c(v_3) < c(v_0)$ and $c(v_3) \leq 3t - 2r$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(C_4) = 3t + 1$.

And, if $3t > 2r$ and $3t \leq 2s$ for $2n = 4$, then using "symmetric replacement" in the previous proof, it can be concluded that $\mathcal{X}_{r,s,t}(C_4) = 3t + 1$.

(2) If **3t** $>$ **2r and 3t** $>$ **2s**, then

if $2n = 3k$, then the following colourings

$$(..., \underline{\mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}, r+1}, \mathbf{1}, ...)$$

and

$$(..., \underline{\mathbf{1}, 2s+1, \mathbf{s{+}1}, 1, \mathbf{2s{+}1}, s+1}, \mathbf{1}, ...)$$

show that $\mathcal{X}_{r,s,t}(C_{2n}) \leq 2r+1$ if $r > s$ and $\mathcal{X}_{r,s,t}(C_{2n}) \leq 2s+1$ if $r \leq s$, respectively, for all $n$ such that $2n = 3k$. On the other hand, lower bounds are given by Observation 4.1 and it follows that if $s < r$, then $\mathcal{X}_{r,s,t}(C_{2n}) = 2r + 1$ and if $r < s$, then $\mathcal{X}_{r,s,t}(C_{2n}) = 2s + 1$, for all n such that $2n = 3k$.

And if $2n \neq 3k$, then the same colouring with $3t+1$ colours as in (1) can be used. Then from Corollary 4.1 it follows that it cannot be done with less colours. Therefore, $\mathcal{X}_{r,s,t}(C_{2n}) = 3t+1$, for all $n$ such that $2n \neq 3k$. □

**Theorem 4.10.** *If $s \leq t \leq r < 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = \begin{cases} s+2t+1 & \text{if} \quad s \leq 2r-2t; \\ 2r+1 & \text{if} \quad s > 2r-2t \text{ and } 2n = 3k; \\ s+2t+1 & \text{if} \quad s > 2r-2t \text{ and } 2n \neq 3k. \end{cases}$$

*Proof. a)* If $\mathbf{s \leq 2r - 2t}$ **and** $(\mathbf{r \geq s+t}$ **or** $\mathbf{2n \geq 5})$, then by Observation 4.1, $\mathcal{X}_{r,s,t}(C_{2n}) \geq s+2t+1$ for all $n \geq 2$. Then the colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{s{+}t{+}1}, s+2t+1}, \mathbf{t{+}1}, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n}) = s+2t+1$, in this situation.

If $s \leq 2r-2t$ and $r < s+t$ for $2n = 4$, suppose $k \leq s+2t$, which is at most $2r$. Then by Lemma 2.29, it may be assumed that $c(v_0), c(v_2) < c(v_1)$. Hence $c(v_0), c(v_2) \leq s+2t-r$ and $r+1 \leq c(v_1) \leq s+2t$. Observe that $s+2t-r > t$. By symmetry, assume that $c(e_1) < c(e_2)$. Then $c(e_1) \leq 2t$ and $c(e_2) \leq s+2t$.

Case 1, $c(e_2) > c(v_1)$: Then $c(v_1) \leq s+t$, $c(e_1) < c(v_1)$ and $c(e_1) \leq s$. Now $c(v_0) > c(e_1)$ implies $c(v_0) \geq t+1$ and on the other hand $c(v_0) < c(v_1)$ implies $c(v_0) \leq s+t-r$, a contradiction.

Case 2, $c(e_2) < c(v_1)$: Then $c(e_2) \leq s+t$ and $c(e_1) \leq t$, which implies $c(v_0) > c(e_1)$, $t+1 \leq c(v_0) \leq s+2t-r$ and $c(e_1) \leq s+t-r$. Now $c(e_4) > c(e_1)$ implies $c(e_4) \geq s+1$, $c(e_4) > c(v_0)$ and $c(e_4) \geq 2t+1$. Next $c(v_3) < c(e_4)$ implies $c(v_3) \leq s+t$, and $c(v_3) > c(v_0)$ implies $c(v_3) \geq r+t+1$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(C_4) \geq s+2t+1$. Furthermore the same colouring as in the last case can be used, hence $\mathcal{X}_{r,s,t}(C_4) = s+2t+1$.

*b)* If $\mathbf{s > 2r - 2t}$, from Observation 4.1 it follows that $\mathcal{X}_{r,s,t}(C_{2n}) \geq 2r+1$. Then, if $2n = 3k$, the colouring

$$(..., \underline{\mathbf{1}, r+1, \mathbf{2r{+}1}, 1, \mathbf{r{+}1}, 2r+1}, \mathbf{1}, ...)$$

shows that $\mathcal{X}_{r,s,t}(C_{2n}) = 2r+1$, for all $n \geq 2$.

But if $2n \neq 3k$, then the colouring with $s+2t+1$ colours used in *a)* can be used too. Then suppose that $k \leq s+2t$. Just three cases are possible from those shown in Corollary 4.1:

9(1), $c(v_0) < c(e_1) < c(e_2) < c(e_3)$: Then $c(v_0) \leq t-s$ and $s+t+1 \leq c(e_2) \leq 2t$. Hence $c(v_1) > c(v_0)$ which implies $c(v_1) \geq r+1$, $c(v_1) > c(e_2)$ and $c(v_1) \geq s+2t+1$, a contradiction.

9(2), $c(v_1) < c(e_1) < c(e_2) < c(e_3)$: Then $c(v_1) \leq t-s$ and $s+t+1 \leq c(e_2) \leq 2t$. Now $c(v_2) > c(v_1)$ implies $c(v_2) \geq r+1$, $c(v_2) > c(e_2)$ and $c(v_2) \geq s+2t+1$, a contradiction.

10, $c(e_1) < c(e_2) < c(e_3) < c(e_4)$: Then $c(e_1) \leq 2t - 2s$ and $s + 1 \leq c(e_2) \leq 2t - s$. If $c(v_0) < c(e_1)$, then $c(v_0) \leq t - 2s$, $t + 1 \leq c(e_1)$ and $s + t + 1 \leq c(e_2)$. Which yields to $c(v_1) > c(v_0)$, $c(v_1) \geq r + 1$, $c(v_1) > c(e_2)$ and $c(v_1) \geq s + 2t + 1$, a contradiction. Hence $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$. Then $c(v_0) > c(e_2)$, which implies $c(v_0) \geq s + t + 1$. If $c(v_1) > c(v_0)$, then $c(v_1) \geq r + s + t + 1 > s + 2t + 1$, a contradiction. Therefore $c(v_1) < c(v_0)$, hence $c(v_1) \leq s + 2t - r$, $c(v_1) < c(e_2)$ and $c(v_1) \leq t - s$. Now from $c(v_1) < c(e_1)$ follows $c(v_1) \leq t - 2s$, $c(e_1) \geq t + 1$ and $c(e_2) \geq s + t + 1$. Then $c(v_2) > c(v_1)$ leads to $c(v_2) > c(e_2)$ and $c(v_2) \geq s + 2t + 1$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(C_{2n}) = s + 2t + 1$, for all $n \geq 2$ and $2n \neq 3k$. □

**Theorem 4.11.** *If $r \leq t \leq s < 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = \begin{cases} r + 2t + 1 & \text{if} \quad r \leq 2s - 2t; \\ 2s + 1 & \text{if} \quad r > 2s - 2t \text{ for } 2n = 3k; \\ r + 2t + 1 & \text{if} \quad r > 2s - 2t \text{ for } 2n \neq 3k. \end{cases}$$

*Proof. a)* If $\mathbf{r \leq 2s - 2t}$ **and** ($\mathbf{s \geq r + t}$ **or** $\mathbf{n \geq 6}$), then by Observation 4.1 it follows that $\mathcal{X}_{r,s,t}(C_{2n}) \geq r + 2t + 1$. From the following colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, r + 2t + 1}, \mathbf{t{+}1}, ...)$$

it can be concluded that $\mathcal{X}_{r,s,t}(C_{2n}) = r + 2t + 1$, for all $n \geq 2$.

If $r \leq 2s - 2t$ and $s < r + t$ for $n = 4$, then the same lower bound could be found using "symmetric replacement" in the corresponding proof of Theorem 4.10. The previous colouring is possible too, hence $\mathcal{X}_{r,s,t}(C_{2n}) = r + 2t + 1$, for all $n \geq 2$.

*b)* For $\mathbf{r > 2s - 2t}$, if $2n = 3k$, Observation 4.1 shows that $\mathcal{X}_{r,s,t}(C_{2n} \geq 2s + 1$ for all $n \geq 3$ and from the colouring

$$(..., \underline{\mathbf{1}, s + 1, \mathbf{2s{+}1}, 1, \mathbf{s{+}1}, 2s + 1}, \mathbf{1}, ...)$$

it follows that $\mathcal{X}_{r,s,t}(C_{2n}) = 2s + 1$, for all $n \geq 3$ such that $2n = 3k$.

If $2n \neq 3k$, then using "symmetric replacement" in Theorem 4.10, it can be concluded that $\mathcal{X}_{r,s,t}(C_{2n}) = s + 2t + 1$, for all $n \geq 2$. □

**Theorem 4.12.** *If $r, s \leq t < r + s$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = \begin{cases} 2t + 1 & \text{if} \quad 2n = 3k; \\ r + s + t + 1 & \text{if} \quad 2n \neq 3k. \end{cases}$$

*Proof. a)* If $\mathbf{2n = 3k}$, then the colouring

$$(..., \underline{\mathbf{t{+}1}, 1, \mathbf{2t{+}1}, t + 1, \mathbf{1}, 2t + 1}, \mathbf{t{+}1}, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n}) \leq 2t+1$ for all $n$, and hence by Observation 4.1, it can be concluded that $\mathcal{X}_{r,s,t}(C_{2n}) = 2t+1$, for all $n \geq 2$.

*b)* If $\mathbf{2n \neq 3k}$, then the colouring

$$(..., \underline{\mathbf{1}, r+t+1, \mathbf{r+1}, r+s+t+1}, \mathbf{1}, ...)$$

shows that $\mathcal{X}_{r,s,t}(C_{2n}) \leq r+s+t+1$.
Then suppose $k \leq r+s+t$. By Corollary 4.1, there are just three possible cases:

Case 9(1), $c(v_0) < c(e_1) < c(e_2) < c(e_3)$: Then $c(v_0) \leq r-s$ and $s+t+1 \leq c(e_2) \leq r+t$. Hence $c(v_1) > c(v_0)$, which implies $c(v_1) \geq r+1$, $c(v_1) > c(e_2)$ and $c(v_1) \geq s+2t+1$, a contradiction.

In case 9(2) it follows a similar contradiction for $c(v_2)$ in relation with $c(v_1)$ and $c(e_2)$.

10, $c(e_1) < c(e_2) < c(e_3) < c(e_4)$: Then $c(e_1) \leq r+t-2s$, $s+1 \leq c(e_2) \leq r+t-s$ and $c(e_3) \leq r+t$. If $c(v_1) < c(e_1)$, then $c(v_1) \leq r-2s$, $t+1 \leq c(e_1)$ and $s+t+1 \leq c(e_2)$. Now $c(v_2) > c(v_1)$ which implies $c(v_2) \geq r+1$, $c(v_2) > c(e_3)$ and $c(v_1) \geq s+2t+1$, a contradiction. Hence $c(v_1) > c(e_1)$, $c(v_1) \geq t+1$, $c(v_1) > c(e_2)$ and $c(v_1) \geq s+t+1$. Next $c(v_2) < c(v_1)$ and $c(v_2) \leq s+t$. Then $c(v_2) < c(e_2)$, which implies $c(v_2) \leq r-s$, $c(e_2) \geq t+1$ and $c(e_3) \geq s+t+1$. Therefore $c(v_3) > c(v_2)$, $c(v_3) \geq r+1$, $c(v_3) > c(e_3)$ and $c(v_2) \geq s+2t+1$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(C_{2n}) = r+s+t+1$, for all $n \geq 2$. □

**Theorem 4.13.** *If $t \geq r+s$, then*

$$\mathcal{X}_{r,s,t}(C_{2n}) = r+s+t+1, \text{ for all } n \geq 2.$$

*Proof.* The result follows as a direct application from Corollary 2 in [11]. □

All results presented in this section are summarized in Table 4.1.

## 4.2 Cycles of odd order

In this section only cycles with an odd number of vertices will be considered.

As shown in Lemmas 2.5 and 2.12, for any cycle of odd order, $C_{2n+1}$, it holds that $\mathcal{X}(C_{2n+1}) = \mathcal{X}'(C_{2n+1}) = \Delta(C_{2n+1}) + 1 = 3$. Hence the bounds given by Lemma 2.25 can be determined.

**Corollary 4.14.** *From Lemma 2.25, it follows that for any cycle of odd order it holds that*

$$\max\{2r+1, 2s+1, t+3\} \leq \mathcal{X}_{r,s,t}(C_{2n+1}) \leq 2r+2s+t+1$$

*for all $n \geq 1$.*

| Conditions | | | $\mathcal{X}_{r,s,t}(C_{2n})$ |
|---|---|---|---|
| $r \geq s + 2t$ | | | r+1 |
| $s \geq r + 2t$ | | | s+1 |
| $(s \leq r < s + t) \wedge (r \geq 2t)$ | | | r+t+1 |
| $(s + t \leq r < s + 2t) \wedge (r \geq 2t)$ | | | s+2t+1 |
| $(r \leq s < r + t) \wedge (s \geq 2t)$ | | | s+t+1 |
| $(r + t \leq s < r + 2t) \wedge (s \geq 2t)$ | | | r+2t+1 |
| $t < r, s < 2t$ | $3t \leq 2r \vee 3t \leq 2s \vee 2n \neq 3k$ | | 3t+1 |
| | $3t > 2r \wedge 3t > 2s \wedge 2n = 3k$ | $r > s$ | 2r+1 |
| | | $r \leq s$ | 2s+1 |
| $s \leq t \leq r < 2t$ | $s \leq 2r - 2t \vee 2n \neq 3k$ | | s+2t+1 |
| | $s > 2r - 2t \wedge 2n = 3k$ | | 2r+1 |
| $r \leq t \leq s < 2t$ | $r \leq 2s - 2t \vee 2n \neq 3k$ | | r+2t+1 |
| | $r > 2s - 2t \wedge 2n = 3k$ | | 2s+1 |
| $r, s \leq t < r + s$ | $2n = 3k$ | | 2t+1 |
| | $2n \neq 3k$ | | r+s+t+1 |
| $t \geq r + s$ | | | r+s+t+1 |

Table 4.1: $[r, s, t]$-chromatic number of even cycles

**Theorem 4.15.** *If $r \geq s + 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r + 1, \text{ for all } n.$$

*Proof.* From the following colouring

$$(..., s + t + 1, \underline{\mathbf{1}, t + 1, \mathbf{r{+}1}, s + t + 1}, \underline{\underline{\mathbf{2r{+}1}, s + 2t + 1}}, \mathbf{1}, t + 1, \mathbf{r{+}1}, ...)$$

it follows that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq 2r + 1$ for all $n$. Then, by the lower bound given by Corollary 4.14, it can be concluded that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r + 1$, for all $n$. ☐

**Theorem 4.16.** *If $s \geq r + 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s + 1, \text{ for all } n.$$

*Proof.* The following colouring

$$(..., r + t + 1, \underline{\mathbf{1}, t + 1, \mathbf{s{+}1}, r + t + 1}, \underline{\underline{\mathbf{2s{+}1}, r + 2t + 1}}, \mathbf{1}, t + 1, \mathbf{s{+}1}, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n}) \leq 2s + 1$ for all $n$. Then, by Corollary 4.14, it follows that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s + 1$, for all $n$. ☐

**Theorem 4.17.** *If $s \leq r < s + 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r + 1, \text{ for all } n.$$

*Proof.* The colouring

$$(..., \max\{s+1,t+1\}, \underline{\max\{\mathbf{2s+t+1}, \mathbf{s+2t+1}\}, \max\{2s+1, s+t+1\}, \mathbf{1},}$$

$$\underline{\max\{s+1,t+1\}}, \underline{\underline{\mathbf{r+1}, 1, \mathbf{2r+1}}}, \max\{2s+1, s+t+1\}, \mathbf{1}, ...)$$

demonstrates that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq 2r+1$ for all $n$. Hence, by Corollary 4.14, $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r+1$ for all $n$. □

**Theorem 4.18.** *If $r \leq s < r+2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s+1, \text{ for all } n.$$

*Proof.* Similarly to the previous theorem, the colouring

$$(..., \max\{r+1,t+1\}, \underline{\max\{\mathbf{2r+t+1}, \mathbf{r+2t+1}\}, \max\{2r+1, r+t+1\}, \mathbf{1},}$$

$$\underline{\max\{r+1,t+1\}}, \underline{\underline{\mathbf{s+1}, 1, \mathbf{2s+1}}}, \max\{2r+1, r+t+1\}, \mathbf{1}, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq 2s+1$ for all $n$. Then by Corollary 4.14, $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s+1$ for all $n$. □

**Theorem 4.19.** *For $t < r, s < 2t$,*

1. *If $2n+1 = 3k$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \begin{cases} 2r+1 & \text{if } \quad s < r; \\ 2s+1 & \text{if } \quad s \geq r. \end{cases}$$

2. *If $2n+1 \neq 3k$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \begin{cases} 2r+1 & \text{if } \quad s < r \text{ and } 2r > 3t; \\ 2s+1 & \text{if } \quad s \geq r \text{ and } 2s > 3t; \\ 3t+1 & \text{otherwise.} \end{cases}$$

*Proof.* (1) If $\mathbf{2n+1} = \mathbf{3k}$, then if $s < r$, the following colouring

$$(..., s+1, \underline{\mathbf{1}, 2r+1, \mathbf{r+1}, 1, \mathbf{2r+1}, s+1}, \mathbf{1}, 2r+1, \mathbf{r+1}, ...)$$

and Corollary 4.14 show that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r+1$ for all $n$.
In a similar way, if $s \leq r$, then from the colouring

$$(..., s+1, \underline{\mathbf{1}, 2s+1, \mathbf{r+1}, 1, \mathbf{2s+1}, s+1}, \mathbf{1}, 2s+1, \mathbf{r+1}, ...)$$

and Corollary 4.14, it follows that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s+1$ for all $n$.

(2) If $\mathbf{2n+1 \neq 3k}$, then if $s < r$, the colouring

$$(..., t+1, \mathbf{1}, \underline{\underline{s+t+1, \mathbf{r+1}, 1, \max\{\mathbf{s+t+1}, \mathbf{2r+1}\}, s+1}},$$

$$\underline{\mathbf{1}, \max\{2s+1, 3t+1\}, \mathbf{2t+1}, t+1}, \mathbf{1}, ...)$$

shows that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq \max\{2s+1, 3t+1\}$ for all $n$. From Corollary 4.14 it follows that, if $2s > 3t$, then $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2s+1$ for all $n$.
And if $3t \geq 2s$, then Corollary 4.1 implies that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 3t+1$ for all $n$.
On the other hand, if $s \geq r$, then the cycle could be coloured as follows

$$(..., \mathbf{t+1}, 1, \underline{\underline{\mathbf{r+t+1}, s+1, \mathbf{1}, \max\{r+t+1, 2s+1\}, \mathbf{r+1}}},$$

$$\underline{1, \max\{\mathbf{2r+1}, \mathbf{3t+1}\}, 2t+1, \mathbf{t+1}}, 1, ...)$$

and applying "symmetric replacement" in the previous case, $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r+1$ for all $n$, if $2r > 3t$, and $\mathcal{X}_{r,s,t}(C_{2n+1}) = 3t+1$ for all $n$, if $3t \geq 2r$. $\square$

**Theorem 4.20.** *If $s \leq t \leq r < 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \begin{cases} 2r+1 & \text{if } 2n+1 = 3k \text{ or } s+2t \leq 2r; \\ s+2t+1 & \text{otherwise.} \end{cases}$$

*Proof. a)* If $\mathbf{2n+1 = 3k}$, then Corollary 4.14 and the colouring

$$(..., \mathbf{r+1}, \underline{1, \mathbf{2r+1}, t+1, \mathbf{1}, r+t+1, \mathbf{r+1}}, 1, \mathbf{2r+1}, t+1, ...)$$

show that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r+1$ for all n.
*b)* If $\mathbf{2n+1 \neq 3k}$, then the colouring

$$(..., t+1, \underline{\mathbf{s+2t+1}, s+t+1, \mathbf{1}, t+1}, \underline{\underline{\mathbf{2r+1}, 1, \mathbf{r+1}, r+t+1}}, \mathbf{1}, t+1, ...)$$

implies that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq \max\{2r+1, s+2t+1\}$. Hence, if $s+2t \leq 2r$, then by Corollary 4.14, $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2r+1$ for all $n$.
Then, if $s+2t > 2r$, suppose $k \leq s+2t$. From Corollary 4.1, there are just three possible cases:

9(1), $c(v_0) < c(e_1) < c(e_2) < c(e_3)$: Then, $s+t+1 \leq c(e_2) \leq 2t$, hence $c(v_2) < c(e_2)$ and $c(v_2) \leq t$. Now $c(v_1) > c(v_2)$ and $c(v_1) \geq r+1$, therefore $c(v_1) \geq c(e_2)$ and $c(v_1) \geq s+2t+1$, a contradiction.

9(2), $c(v_1) < c(e_1) < c(e_2) < c(e_3)$: Then $c(v_1) \leq t-s$ and $s+t+1 \leq c(e_2) \leq 2t$. Hence $c(v_2) > c(v_1)$ and $c(v_2) \geq r+1$ and, on the other hand, $c(v_2) < c(e_2)$ and $c(v_2) \leq t$, a contradiction.

10, $c(e_1) < c(e_2) < c(e_3) < c(e_4)$: Then $c(e_1) \leq 2t-2s$, $s+1 \leq c(e_2) \leq 2t-s$ and $c(e_3) \leq 2t$. If $c(v_1) < c(e_1)$ (only possible if $t > 2s$), then $c(v_1) \leq t-2s$, $c(e_1) \geq t+1$ and $c(e_2) \geq s+t+1$. Then $c(v_2) > c(v_1)$, $c(v_2) > c(e_2)$ and $c(v_2) \geq s+2t+1$, a contradiction. Hence $c(v_1) > c(e_1)$. Then $c(v_1) > c(e_2)$ and $c(v_1) \geq s+t+1$.

Therefore $c(v_2) < c(v_1)$, $c(v_2) < c(e_2)$ and $c(v_2) \leq t - s$. Now $c(e_2) \geq t + 1$ and $c(e_3) \geq s + t + 1$. Hence $c(v_3) > c(v_2)$, $c(v_3) > c(e_3)$ and $c(v_3) \geq s + 2t + 1$, a contradiction.

Hence, $\mathcal{X}_{r,s,t}(C_{2n+1}) = s + 2t + 1$ for all $n$. ☐

**Theorem 4.21.** *If $r \leq t \leq s < 2t$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \begin{cases} 2s+1 & \text{if } 2n+1 = 3k \text{ or } r+2t \leq 2s; \\ r+2t+1 & \text{otherwise.} \end{cases}$$

*Proof.* The proof follows directly using "symmetric replacement" in Theorem 4.20. And the colourings used would be

$$(..., s+1, \underline{\mathbf{1}, 2s+1, \mathbf{t+1}, 1, \mathbf{s+t+1}, s+1}, \mathbf{1}, 2s+1, \mathbf{t+1}, ...)$$

if $2n + 1 = 3k$, and

$$(..., \mathbf{t+1}, \underline{r+2t+1, \mathbf{r+t+1}, 1, \mathbf{t+1}}, \underline{\underline{2s+1, \mathbf{1}, s+1, \mathbf{s+t+1}}}, 1, \mathbf{t+1}, ...)$$

if $2n + 1 \neq 3k$. ☐

**Theorem 4.22.** *If $r, s \leq t < r + s$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \begin{cases} 2t+1 & \text{if } 2n+1 = 3k; \\ r+s+t+1 & \text{otherwise.} \end{cases}$$

*Proof. a)* If $\mathbf{2n+1=3k}$, then the colouring

$$(..., t+1, \underline{\mathbf{1}, 2t+1, \mathbf{t+1}, 1, \mathbf{2t+1}, t+1}, \mathbf{1}, 2t+1, ...)$$

and Observation 4.1 show that $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2t + 1$, for all $n$.
*b)* If $\mathbf{2n+1 \neq 3k}$, from the colouring

$$(..., r+s+t+1, \underline{\mathbf{r+1}, r+t+1, \mathbf{1}, r+s+t+1}, \underline{\underline{\mathbf{t+1}, 1, \mathbf{r+s+t+1}, r+s+1,}}$$
$$\mathbf{1}, r+s+t+1, ...)$$

it follows that $\mathcal{X}_{r,s,t}(C_{2n+1}) \leq r + s + t + 1$, for all $n$.
Suppose $k \leq r + s + t$. Then, by Corollary 4.1, there are just four possible cases:

1, $c(v_0) < c(v_1) < c(v_2) < c(v_3)$: Then $c(v_0) \leq s+t-2r$ and $r+1 \leq c(v_1) \leq s+t-r$. If $c(e_1) < c(v_0)$ (only if $s > 2r$), then $c(e_1) \leq s-2r$, $c(v_0) \geq t+1$ and $c(v_1) \geq r+t+1$. Therefore $c(e_2) > c(e_1)$, $c(e_2) \geq s + 1$, $c(e_2) > c(v_1)$ and $c(e_2) \geq r + 2t + 1$, a contradiction. Hence $c(e_1) > c(v_0)$ and $c(e_1) \geq t + 1$. Then $c(e_1) > c(v_1)$, $c(e_1) \geq r + t + 1$ and $c(v_1) \leq r + s$. Now $c(e_2) < c(e_1)$, $c(e_2) \leq r + t$, $c(e_2) < c(v_1)$, $c(e_2) \leq s-r$ and $c(v_2) \geq r+t+1$. Therefore $c(e_3) > c(e_2)$, $c(e_3) \geq s+1$, $c(e_3) > c(v_2)$ and $c(e_3) \geq r + 2t + 1$, a contradiction.

2(1), $c(v_0) < c(v_1) < c(v_2) < c(e_3)$: Then $r+1 \leq c(v_1) \leq s$. Hence $c(e_1) > c(v_1)$, $c(e_1) \geq r+t+1$, $c(e_2) < c(e_1)$ and $c(e_2) \leq r+t+1$ (this implies that case 2(2) does not have to be considered). On the other hand $c(e_2) > c(v_1)$ and $c(e_2) \geq r+t+1$, a contradiction.

Case 9 and case 10 lead also to a contradiction applying "symmetric replacement" in cases 1 and 2.

Hence, $\mathcal{X}_{r,s,t}(C_{2n+1}) = r+s+t+1$, for all $n$. □

**Theorem 4.23.** *If $t \geq r+s$, then*

$$\mathcal{X}_{r,s,t}(C_{2n+1}) = \min\{2t+1, \max\{2r+s+t+1, r+2s+t+1\}\}.$$

*Proof.* If there exists a monotone sequence of colours of alternating elements of the cycle of length three (this is $c(v_i) < c(e_{i+1}) < c(v_{i+1})$ or $c(e_i) < c(v_i) < c(e_{e+1})$), then $k \geq 2t+1$.
In the other case, if $x$ is an arbitrary element of $G$, then $c(x) < c(y)$ for all elements $y$ that are incident to $x$ or $c(x) > c(y)$ for all $y$. Using induction, this implies that either $c(v) < c(e)$ for all vertices $v$ and all edges $e$ or always $c(v) > c(e)$. Without loss of generality, let $c(v) < c(e)$. It also holds that $c(v_{i-1}) < c(v_i) < c(v_{i+1})$ and $c(e_{j-1}) < c(e_j) < c(e_{j+1})$ for some $i$ and $j$ (or symmetric situations) because of the parity of the cycle. Suppose that $c(v_0) < c(v_1) < c(v_2)$, then $c(e_2) \geq 2r+t+1$. If $c(e_3) > c(e_2)$, then $c(e_3) \geq 2r+s+t+1$. In the other case, it holds $c(e_3) \geq 2r+t+1$, hence $c(e_2) \geq 2r+s+t+1$. Similarly if the case is $c(e_{j-1}) < c(e_j) < c(e_{j+1})$, then it can be proved that $k \geq r+2s+t+1$.
Therefore, $k \geq \min\{2t+1, \max\{2r+s+t+1, r+2s+t+1\}\}$.

Then, if $t \leq \max\{2r+s, r+2s\}$ for $2n+1 = 3k$, the cycle could be coloured like

$$(..., t+1, \underline{\mathbf{1}, 2t+1, \mathbf{t+1}, 1, \mathbf{2t+1}, t+1}, \mathbf{1}, 2t+1, ...),$$

and for $2n+1 \neq 3k$

$$(..., \mathbf{1}, \underline{t+1, \mathbf{2t+1}, 1, \mathbf{t+1}, 2t+1, \mathbf{1}}, \underline{\underline{r+t+1, \mathbf{r+1}, r+s+t+1}}, \mathbf{1}, t+1, ...).$$

Hence if $t \leq \max\{2r+s, r+2s\}$, then $\mathcal{X}_{r,s,t}(C_{2n+1}) = 2t+1$ for all $n$.

On the other hand, if $t > \max\{2r+s, r+2s\}$, then the colourings

$$(..., \mathbf{r+s+t+1}, \underline{1, \mathbf{s+t+1}, s+1, \mathbf{r+s+t+1}}, \underline{\underline{2s+1, \mathbf{2r+s+t+1}}}, 1, ...)$$

if $r \geq s$, and

$$(..., r+s+t+1, \underline{\mathbf{1}, r+t+1, \mathbf{r+1}, r+s+t+1}, \underline{\underline{\mathbf{2r+1}, r+2s+t+1}}, \mathbf{1}, ...)$$

if $r < s$, show that $\mathcal{X}_{r,s,t}(C_{2n+1}) = \max\{2r+s+t+1, r+2s+t+1\}$ for all n. □

All results presented in this section are summarized in Table 4.2.

| Conditions | | | $\mathcal{X}_{r,s,t}(C_{2n+1})$ |
|---|---|---|---|
| $r \geq s + 2t$ | | | 2r+1 |
| $s \geq r + 2t$ | | | 2s+1 |
| $(s \leq r < s + 2t) \wedge (r \geq 2t)$ | | | 2r+1 |
| $(r \leq s < r + 2t) \wedge (s \geq 2t)$ | | | 2s+1 |
| $t < r, s < 2t$ | $2n + 1 = 3k$ | $s < r$ | 2r+1 |
| | | $r \leq s$ | 2s+1 |
| | $2n + 1 \neq 3k$ | $r > s \wedge 2r > 3t$ | 2r+1 |
| | | $r \leq s \wedge 2s > 3t$ | 2s+1 |
| | | otherwise | 3t+1 |
| $s \leq t \leq r < 2t$ | $2n + 1 = 3k$ | | 2r+1 |
| | $2n + 1 \neq 3k$ | $s + 2t \leq 2r$ | 2r+1 |
| | | $s + 2t > 2r$ | s+2t+1 |
| $r \leq t \leq s < 2t$ | $2n + 1 = 3k$ | | 2s+1 |
| | $2n + 1 \neq 3k$ | $r + 2t \leq 2s$ | 2s+1 |
| | | $r + 2t > 2s$ | r+2t+1 |
| $r, s \leq t < r + s$ | $2n + 1 = 3k$ | | 2t+1 |
| | $2n + 1 \neq 3k$ | | r+s+t+1 |
| $t \geq r + s$ | $t \leq \max\{2r + s, r + 2s\}$ | | 2t+1 |
| | $t > 2r + s \geq r + 2s$ | | 2r+s+t+1 |
| | $t > r + 2s > 2r + s$ | | r+2s+t+1 |

Table 4.2: $[r, s, t]$-chromatic number of odd cycles

# Chapter 5

# $[r, s, t]$-Colouring of Stars

In this chapter the $[r, s, t]$-colouring of the stars is studied, which is of special interest since, by Lemma 2.22, the $[r, s, t]$-chromatic number of the star with $\Delta(G)$ leaves is a lower bound for the $[r, s, t]$-chromatic number of the whole graph $G$.

## 5.1 Improvement of the general bounds

In Lemma 2.25, it was found a first improvement of the general bounds given by Kemnitz and Marangio [11], considering the star containing a vertex of maximum degree and its neighbors, but only the colours of the central vertex and the edges were studied. Obviously the more elements are taken, the better are the bounds. Working in this direction, a new improvement of the general bounds has been found.

**Lemma 5.1.** *For the [r,s,t]-chromatic number of a graph G, it holds*

$$\max\{r(\mathcal{X}(G)-1)+1, s(\mathcal{X}'(G)-1)+1, \min\{r+s(\Delta-1)+t+1, s(\Delta-2)+2t+1, \max\{r+1, 2t+1, s(\Delta-1)+t+1\}\}\} \leq \mathcal{X}_{r,s,t}(G) \leq r(\mathcal{X}(G)-1)+s(\mathcal{X}'(G)-1)+t+1,$$

*if* $|V(G)| \geq 2$ *and* $G \neq N_n$ *(where* $\Delta(G)$ *is the maximum degree of* $G$ *and* $N_n$ *is the empty graph).*

*Proof.* From these bounds, just the third term of the maximum in the lower bound has not been proved. This partial proof will be now given.

Consider the star $K_{1,\Delta}$ consisting of a vertex, $v_0$, of maximum degree $\Delta := \Delta(G)$ and its adjacent vertices without other edges than the ones connecting $v_0$ and its neighbors. It needs at least the following number of colours: To colour the $\Delta$ edges at least $s(\Delta-1)+1$ colours have to be used. The colour of the "central vertex" has the following possibilities in the different cases.

If its colour "fits" between the colours of two edges, then the difference between these two colours must be at least $2t$. For the $m$ edges with smaller colour than that

of the central vertex, $s(m-1)+1$ colours are needed, and for the $\Delta - m$ edges with larger colour, $s(\Delta - m - 1) + 1$ have to be at least used. Hence the total number of colours used to colour the star is at least $s(\Delta - 2) + 2t + 1$.
On the other hand if the colour of the central vertex is smaller than the smallest one for the edges or greater than the greatest, then at least $t + s(\Delta - 1) + 1$ colours are being used. Suppose that it is greater (the other case is symmetric). It means $c(v_0) > c(e_1)$, where $v_0$ is the central vertex and $e_1$ is such that $c(e_1) = \max\{c(e_i) : e_i$ is an edge of the considered $K_{1,\Delta}\}$. Then, if the colour of the other end-vertex of $e_1$, $v_1$, is greater than $c(e_1)$, at least $r + s(\Delta - 1) + t + 1$ colours should be used. In the other case, if $c(v_1)$ is smaller than $c(e_1)$, then the lower bound would be $\max\{r + 1, 2t + 1, s(\Delta - 1) + t + 1\}$ colours.

Then, by Lemma 2.22, $\min\{r + s(\Delta - 1) + t + 1, s(\Delta - 2) + 2t + 1, \max\{r + 1, 2t + 1, s(\Delta - 1) + t + 1\}\} \leq \mathcal{X}_{r,s,t}(K_{1,\Delta}) \leq \mathcal{X}_{r,s,t}(G)$.

The other terms in the inequality were already proved in Lemma 2.25. $\square$

This new bound will be very useful in the study of the $[r, s, t]$-chromatic number of stars, since it will be often sharp.

## 5.2 $K_{1,3}$

As a first approach to the problem of finding the value of the $[r, s, t]$-chromatic number of $K_{1,n}$ for any $n$, this task will be done for stars with three leaves.

**Notation 3.** *In this section the notation shown in Figure 5.1 for the vertices and edges of $K_{1,3}$ will be used.*

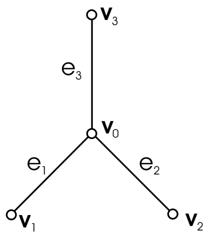



Figure 5.1: Notation of $K_{1,3}$

Since a star is a bipartite graph, $\mathcal{X}(K_{1,3}) = 2$ and $\mathcal{X}'(K_{1,3}) = \Delta(K_{1,3}) = 3$, as shown in Lemma 2.6 and Theorem 2.11. Using this fact, from Lemma 5.1, the next Corollary follows.

**Corollary 5.2.** *For a star $K_{1,3}$, it holds*

$\max\{r + 1, 2s + 1, \min\{r + 2s + t + 1, s + 2t + 1, \max\{r + 1, 2t + 1, 2s + t + 1\}\}\} \leq$
$\leq \mathcal{X}_{r,s,t}(K_{1,3}) \leq r + 2s + t + 1$.

As shown in Lemma 2.19, the total-chromatic number of $K_{1,3}$ is 4. This fact implies that there is a monotone sequence of colours of four elements of the star so that each element is in contact with those elements whose colours are the immediately precedent and the immediately following of its colour in the sequence. This property will be very useful in order to find the $[r, s, t]$-chromatic number of $K_{1,3}$ and it is presented in the following Corollary.

**Corollary 5.3.** *For any triple $[r, s, t]$ one of the following constellations occurs (reduction made because of Lemma 2.29), which leads to the corresponding bounds:*

**1.** *If* $c(v_1) \leq c(v_0) \leq c(v_2) \leq c(e_2)$*, then* $k \geq 2r + t + 1$.
**2.** *If* $c(v_1) \leq c(v_0) \leq c(e_2) \leq c(v_2)$*, then* $k \geq \max\{r + 2t + 1, 2r + 1\}$.
**3.** *If* $c(v_1) \leq c(v_0) \leq c(e_2) \leq c(e_3)$ *(**3(1)**) or*
$c(v_1) \leq c(v_0) \leq c(e_2) \leq c(e_1)$ *(**3(2)**) or*
$c(v_1) \leq c(v_0) \leq c(e_1) \leq c(e_2)$ *(**3(3)**), then* $k \geq r + s + t + 1$.
**4.** *If* $c(v_1) \leq c(e_1) \leq c(v_0) \leq c(e_3)$*, then* $k \geq \max\{3t + 1, r + t + 1, s + t + 1\}$.
**5.** *If* $c(e_1) \leq c(e_2) \leq c(e_3) \leq c(v_3)$ *(**5(1)**) or*
$c(e_1) \leq c(e_2) \leq c(e_3) \leq c(v_0)$ *(**5(2)**), then* $k \geq 2s + t + 1$.

Using the bounds given by the previous Corollaries, the $[r, s, t]$-chromatic number of the stars will be determined for each case.

**Theorem 5.4.** *If* $r \geq s + 2t$*, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} r+1 & \text{if } \; r \geq 2s+2t; \\ 2s+2t+1 & \text{if } \; 2s+t \leq r < 2s+2t; \\ r+t+1 & \text{if } \; 2s \leq r < 2s+t; \\ 2s+t+1 & \text{if } \; r < 2s \leq r+t; \\ r+2t+1 & \text{if } \; r+t < 2s \leq r+2t; \\ 2s+1 & \text{if } \; 2s > r+2t. \end{cases}$$

*Proof.* For $r \geq s + 2t$ the colouring given by Figure 5.2 proves that if $r \geq 2s + 2t$,

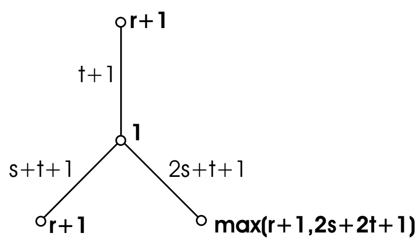



Figure 5.2: $r \geq s + 2t$

then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r + 1$ and if $r < 2s + 2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s + 2t + 1$. Hence if $r \geq 2s + 2t$, by Corollary 5.2 $\mathcal{X}_{r,s,t}(K_{1,3}) = r + 1$.

If $r < 2s+2t$, suppose that $k \leq 2s+2t$. By Corollary 5.3 there are just two possible situations:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $c(e_1) \leq t$ and $2s+t+1 \leq c(v_3) \leq 2s+2t$. Therefore $c(v_0) < c(v_3)$, so $c(v_0) \leq 2s+2t-r$. Furthermore $c(e_1) < c(v_0)$ and $c(v_0) \geq t+1$, a contradiction if $r \geq 2s+t$.

Case 5(2) leads to an analogous contradiction considering $c(v_1)$, $r \geq 2s+t$.

Hence if $2s+t \leq r < 2s+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+2t+1$.

Then, if $r < 2s+t$, a colouring with less colours can be found (see Figure 5.3), hence $\mathcal{X}_{r,s,t}(K_{1,3}) \leq \max\{r+t+1, 2s+t+1\}$.

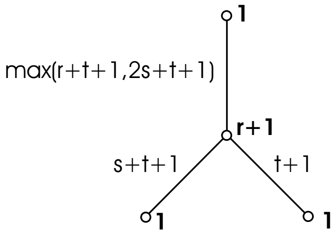



Figure 5.3: $s+2t \leq r < 2s+t$

If $r \geq 2s$ suppose $k \leq r+t$. By Corollary 5.3 just two situations are possible:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $c(e_1) \leq r-2s < t$ and $2s+t+1 \leq c(v_3) \leq r+t$. Hence $c(v_0) < c(v_3)$, so $c(v_0) \leq t$. Then $c(e_1) > c(v_0)$, which implies $c(e_1) \geq t+1$, a contradiction.

From case 5(2) a similar contradiction follows considering $c(v_1)$ in relation with $c(v_0)$ and $c(e_1)$.

Hence if $2s \leq r < 2s+t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r+t+1$.

If $r < 2s$, by the previous colouring $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+t+1$. Suppose $k \leq 2s+t$. By Corollary 5.3 there is just one possible situation:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t+1 \leq c(e_1) \leq s+t$, $r+1 \leq c(v_0) \leq 2s$ and $r+t+1 \leq c(e_3) \leq 2s+t$. Hence $c(v_2) < c(v_0)$ and $c(v_2) \leq 2s-r$. On the other hand, $c(e_2) < c(e_3)$ and $c(e_2) \leq s+t$, hence $c(e_2) < c(e_1)$ and $c(e_2) \leq t$. So $c(v_2) > c(e_2)$ and $c(v_2) \geq t+1$, which is a contradiction if $2s \leq r+t$.

Hence, if $r < 2s \leq r+t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+t+1$.

Finally, if $r \geq s+2t$ and $2s > r+t$ the colouring shown in Figure 5.4 is possible.

This implies that in this case if $2s \leq r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r+2t+1$, and if $2s > r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+1$. In the last case, by Corollary 5.2 the inequality becomes equality, hence if $2s > r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+1$.
And if $2s \leq r+2t$ by Corollary 5.3 there is just one possible situation:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t+1 \leq c(e_1) \leq r+2t-s$, $r+1 \leq c(v_0) \leq r+t$ and $r+t+1 \leq c(e_3) \leq r+2t$. Hence, $c(v_2) < c(v_0)$ and $c(v_2) \leq t$. Furthermore

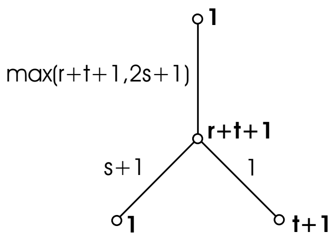



Figure 5.4: $r \geq s + 2t$ and $2s > r + t$

$c(e_2) < c(e_3)$ and $c(e_2) \leq r + 2t - s$, so $c(e_2) < c(e_1)$ and $c(e_2) \leq r + 2t - 2s$ which is smaller than $t$. Therefore $c(v_2) > c(e_2)$ and $c(v_2) \geq t + 1$, a contradiction.

Hence if $r + t < 2s \leq r + 2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r + 2t + 1$. □

**Theorem 5.5.** *If $s + t \leq r < s + 2t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} 2s + 2t + 1 & \text{if} \quad r > 2s + t; \\ r + t + 1 & \text{if} \quad 2s \leq r \leq 2s + t; \\ 2s + t + 1 & \text{if} \quad r < 2s \leq r + t; \\ 2s + 1 & \text{if} \quad 2s > r + t. \end{cases}$$

*Proof. a)* If $\mathbf{r} > \mathbf{2s} + \mathbf{t}$, then the colouring given by Figure 5.5 shows that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s + 2t + 1$.

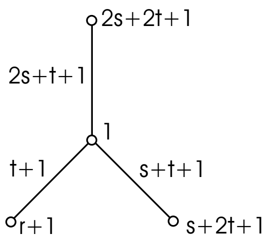



Figure 5.5: $2s + t \leq r < s + 2t$, $r \geq 2t$

Suppose $k \leq 2s + 2t$. The only possible situations by Corollary 5.3 are the following:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $c(e_1) \leq t$ and $2s + t + 1 \leq c(v_3) \leq 2s + 2t$. Hence $c(v_0) < c(v_3)$ and $c(v_0) \leq 2s + 2t - r$ which is smaller than $t$. Therefore, $c(e_1)$ should be greater than $c(v_0)$, but this is not possible.

Case 5(2) leads to an analogous contradiction, considering $c(v_1)$ in relation with $c(v_0)$ and $c(e_1)$.

Hence, $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s + 2t + 1$.

*b)* If $\mathbf{2s} \leq \mathbf{r} \leq \mathbf{2s} + \mathbf{t}$, then the colouring given in Figure 5.6 proves that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r + t + 1$.
Suppose that $k \leq r + t$. Then by Corollary 5.3 just two cases are possible:

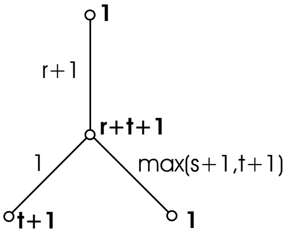



Figure 5.6: $s+t \leq r < s+2t$, $r \geq 2t$ and $2s \leq r \leq 2s+t$

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $c(e_1) \leq r-2s$ and $2s+t+1 \leq c(v_3) \leq r+t$. Hence, $c(v_0) < c(v_3)$ and $c(v_0) \leq t$. Therefore $c(e_1) > c(v_0)$ and $c(e_1) \geq t+1$, a contradiction.

From case 5(2) follows an analogous contradiction, considering $c(v_1)$.

Hence, $\mathcal{X}_{r,s,t}(K_{1,3}) = r+t+1$.

*c)* If $\mathbf{r} < \mathbf{2s}$, then from the colouring shown by Figure 5.7, it follows that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+t+1$.

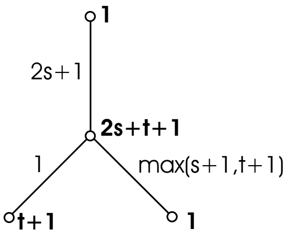



Figure 5.7: $s+t \leq r < s+2t$ and $2t \leq r < 2s$

Then by Corollary 5.3 there is just one possible situation:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $r+1 \leq c(v_0) \leq 2s$. Therefore $c(v_2) < c(v_0)$ and $c(v_2) \leq 2s-r$. If $c(e_2) > c(v_2)$, then $c(e_2) \geq t+1$ and $\max\{c(e_i) : i = 1, ..., 3\} \geq 2s+t+1$, a contradiction. Hence $c(e_2) < c(v_2)$, and then $c(e_2) \leq 2s-r-t$, which is only possible if $2s > r+t$.

Hence if $2s \leq r+t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+t+1$.

If $2s > r+t$, then Figure 5.8 implies that $K_{1,3}$ can be coloured with less colours. Hence, $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+1$ and, by Corollary 5.2, $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+1$.

$\square$

**Theorem 5.6.** *If $s \leq r < s+t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} 2s+1 & \text{if} \quad 2s \geq r+2t; \\ r+2t+1 & \text{if} \quad r+t \leq 2s < r+2t; \\ 2s+t+1 & \text{otherwise.} \end{cases}$$

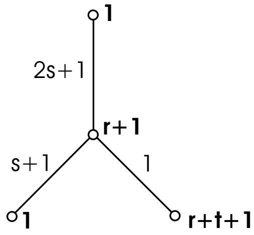



Figure 5.8: $s + t \leq r < s + 2t$, $r \geq 2t$ and $2s > r + t$

*Proof.* For $s \leq r < s + t$ and $r \geq 2t$ the colouring given in Figure 5.9 shows that if $2s \geq r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+1$ and if $2s < r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r+2t+1$.

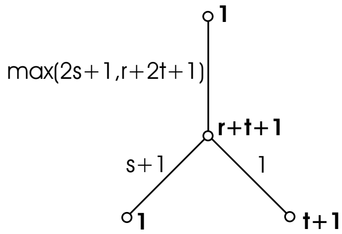



Figure 5.9: $s \leq r < s + t$, $r \geq 2t$ and $2s \geq r + 2t$

Then in the first case by Corollary 5.2, $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s + 1$.
On the other hand if $2s < r + 2t$, supposed that $k \leq r + 2t$, by Corollary 5.3 there is just one possible situation if $2s \geq r + t$:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t+1 \leq c(e_1) \leq r+2t+1$, $r+1 \leq c(v_0) \leq r+t$ and $r + t + 1 \leq c(e_3) \leq r + 2t$. Therefore $c(e_2) < c(e_3)$ and $c(e_2) \leq r + 2t - s$, so $c(e_2) < c(e_1)$ and $c(e_2) \leq r + 2t - 2s$ which is smaller than $t$. Hence $c(v_2) > c(e_2)$ and $c(v_2) \geq t + 1$, that implies $c(v_2) > c(v_0)$ and $c(v_2) \geq 2r + 1$, a contradiction.

Hence if $r + t \leq 2s < r + 2t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r + 2t + 1$.

Then if $2s < r + t$, a colouring with less colours can be found (see Figure 5.10), so

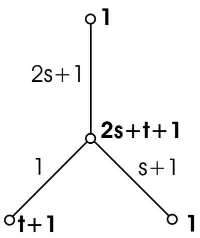



Figure 5.10: $s \leq r < s + t$, $r \geq 2t$ and $2s < r + t$

that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s + t + 1$. Suppose $k \leq 2s + t$, then by Corollary 5.3 the only possible situation is:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t+1 \leq c(e_1) \leq s+t$, $r+1 \leq c(v_0) \leq 2s$ and $r+t+1 \leq c(e_3) \leq 2s+t$. Hence $c(e_2) < c(e_3)$ and $c(e_2) \leq s+t$, so $c(e_2) < c(e_1)$ and $c(e_2) < t$. Therefore $c(v_2) > c(e_2)$ and $c(v_2) \geq t+1$, and then $c(v_2) > c(v_0)$ and $c(v_2) \geq 2r+1$ which is greater than $2s+t$, a contradiction.

Hence $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+t+1$. □

**Theorem 5.7.** *If $s \geq r$ and $s \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+1$$

*Proof.* From the colouring shown in Figure 5.11 it follows that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+1$.

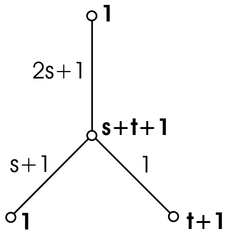



Figure 5.11: $s \geq r$ and $s \geq 2t$

Then by Corollary 5.2, $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+1$. □

**Theorem 5.8.** *If $t < r, s < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} r+2t+1 & \text{if} \quad r \geq s \text{ and } 2s \geq r+t\ ; \\ 2s+t+1 & \text{if} \quad r \geq s \text{ and } 2s < r+t; \\ s+2t+1 & \text{otherwise.} \end{cases}$$

*Proof. a)* If $\mathbf{r} \geq \mathbf{s}$, then from the colouring shown by Figure 5.12, it follows that

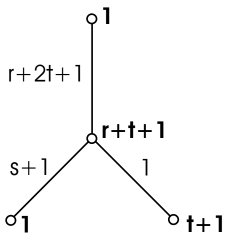



Figure 5.12: $t < r \geq s < 2t$

$\mathcal{X}_{r,s,t}(K_{1,3}) \leq r+2t+1$. If supposed that $k \leq r+2t$, by Corollary 5.3 the only possible situations are:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t+1 \leq c(e_1) \leq r$, $2t+1 \leq c(v_0) \leq r+t$ and $3t+1 \leq c(e_3) \leq r+2t$. Hence $c(e_2) < c(e_3)$ and therefore $c(e_2) < c(v_0)$ and $c(e_2) \leq r$, so $c(e_2) < c(e_1)$ and $c(e_2) \leq r - s$. Therefore $c(v_2) > c(e_2)$ and $c(v_2) \geq t+1$, but then $c(v_2)$ must be greater than $c(v_0)$, which is a contradiction.

Case 5(1) and case 5(2) are just possible if $2s < r + t$.

Hence if $2s \geq r + t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r + 2t + 1$.

On the other hand, if $2s < r+t$, then the colouring given by Figure 5.13 proves that at most $2s + t + 1$ colours are needed.

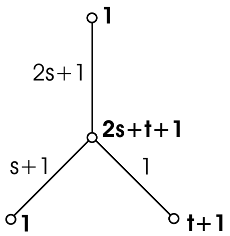



Figure 5.13: $t < s \leq r < 2t$ and $2s < r + t$

Suppose $k \leq 2s + t$, then by Corollary 5.3 just one situation is possible:

4, $c(v_1) < c(e_1) < c(v_0) < c(e_3)$: Then $t + 1 \leq c(e_1) \leq 2s - t$ and $3t + 1 \leq c(e_3) \leq 2s + t$. Observe that it is not possible that $c(e_2)$ and $c(e_3)$ are greater than $c(e_1)$, because in other case at least $2s+t+1$ colours would be needed. Hence $c(e_2) < c(e_1)$ and $c(e_2) \leq s - t$, so $c(v_2) > c(e_2)$ and $c(v_2) \geq t + 1$. Then $c(v_2)$ should be greater than $c(v_0)$, a contradiction.

Hence if $2s < r + t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s + t + 1$.

*b)* If $\mathbf{r} < \mathbf{s}$, then the colouring given by Figure 5.14 implies that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq s + 2t + 1$ which coincides with the lower bound given by Corollary 5.2.

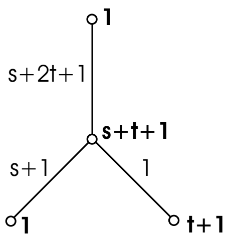



Figure 5.14: $t < r < s < 2t$

Hence $\mathcal{X}_{r,s,t}(K_{1,3}) = s + 2t + 1$. □

**Theorem 5.9.** *If $r \leq t \leq s < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = s + 2t + 1.$$

*Proof.* From the colouring in Figure 5.15 follows an upper bound that coincides with the lower bound given by Corollary 5.2.
Hence, $\mathcal{X}_{r,s,t}(K_{1,3}) = s + 2t + 1$. □

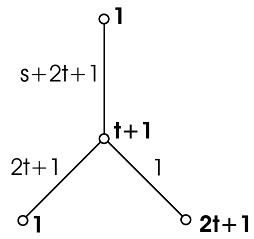



Figure 5.15: $r \leq t \leq s < 2t$

**Theorem 5.10.** *If $s \leq t \leq r < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} s + 2t + 1 & \textit{if} \quad s + t \geq r; \\ r + t + 1 & \textit{if} \quad s + t < r \leq 2s + t; \\ 2s + 2t + 1 & \textit{otherwise.} \end{cases}$$

*Proof.* The colouring given by Figure 5.16 shows that if $s+t \geq r$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq s + 2t + 1$ and if $s + t < r$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r + t + 1$.

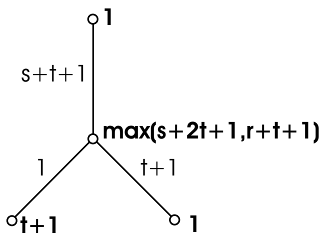



Figure 5.16: $s \leq t \leq r < 2t$

Then if $s + t \geq r$, supposed that $k \leq s + 2t$, by Corollary 5.3 there are just two possibilities:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $s + 1 \leq c(e_2) \leq t$ and $2s + t + 1 \leq c(v_3) \leq s + 2t$. Hence $c(e_2) < c(v_0) < c(v_3)$, so $s + t + 1 \leq c(v_0) \leq s + 2t - r$, a contradiction.

Case 5(2) leads to a similar contradiction considering $c(v_2)$.

Hence if $s + t \geq r$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = s + 2t + 1$.

On the other hand if $s + t < r$, suppose that $k \leq r + t$, then there are just two possibilities left by Corollary 5.3:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $2s + t + 1 \leq c(v_3) \leq r + t$, so $c(v_0) < c(v_3)$ and $c(v_0) \leq t$. Therefore $c(e_1) > c(v_0)$, so $c(e_1) \geq t + 1$ and $c(v_3) \geq 2s + 2t + 1$, a contradiction if $2s + t \geq r$.

From case 5(2) it follows a similar contradiction considering $c(v_1)$, if $2s + t \geq r$.

Hence if $s + t < r \leq 2s + t$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r + t + 1$.

Then if $2s + t \le r$ a colouring using less colours is possible (see Figure 5.17) and $\mathcal{X}_{r,s,t}(K_{1,3}) \le 2s + 2t + 1$.

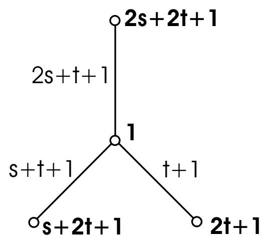



Figure 5.17: $2s + t \le r < 2t$

Suppose $k \le 2s + 2t$, then by Corollary 5.3 just two cases are possible:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $c(e_1) \le t$ and $2s + t + 1 \le c(v_3) \le 2s + 2t$. Therefore $c(e_1) < c(v_0) < c(v_3)$, so $t + 1 \le c(v_0) \le 2s + 2t - r$, a contradiction.

5(2) leads to a similar contradiction considering $c(v_1)$.

Hence if $2s + t \le r$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s + 2t + 1$.

□

**Theorem 5.11.** *If $r, s \le t < r + s$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} r + s + t + 1 & \textit{if} \quad r \ge s; \\ 2s + t + 1 & \textit{otherwise.} \end{cases}$$

*Proof.* From the colouring given by Figure 5.18, it follows that if $r \ge s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \le r + s + t + 1$ and if $r < s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \le 2s + t + 1$.

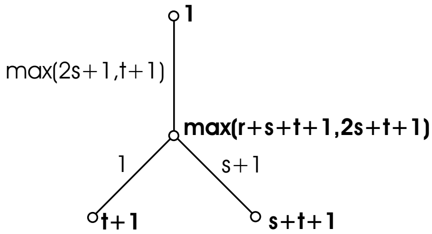



Figure 5.18: $r, s \le t < r + s$

In the second case, the upper bound coincides with the lower bound given by Corollary 5.2, hence $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+t+1$ if $r < s$. On the other hand, if $r \ge s$, suppose $k \le r + s + t$. Then by Corollary 5.3 just two situations are possible:

5(1), $c(e_1) < c(e_2) < c(e_3) < c(v_3)$: Then $s + 1 \le c(e_2) \le r$ and $2s + t + 1 \le c(v_3) \le r+s+t$. Therefore $c(e_2) < c(v_0) < c(v_3)$, so $s+t+1 \le c(v_0) \le s+t$, a contradiction.

From case 5(2) it follows a similar contradiction, considering $c(v_2)$.

Hence if $r \ge s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = r + s + t + 1$. □

**Theorem 5.12.** *If $t \geq r+s$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = \begin{cases} r+2s+t+1 & \textit{if} \quad t \geq r+2s; \\ 2t+1 & \textit{if} \quad 2s \leq t < r+2s; \\ 2s+t+1 & \textit{otherwise.} \end{cases}$$

*Proof. a)* If $t \geq r+2s$, then the colouring given by Figure 5.19 demonstrates that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq r+2s+t+1$. This upper bound coincides with the lower bound given by Corollary 5.2.

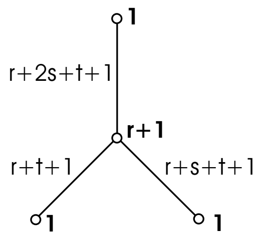



Figure 5.19: $t \geq r+2s$

Hence, $\mathcal{X}_{r,s,t}(K_{1,3}) = r+2s+t+1$.

*b)* If $t < r+2s$, then from the colouring shown by Figure 5.20, it follows that if $t \geq 2s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2t+1$ and if $t < 2s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s+t+1$.

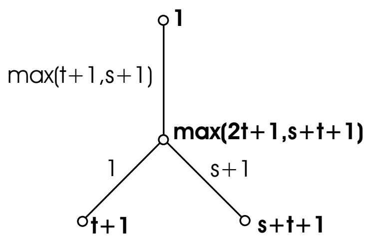



Figure 5.20: $r+s \leq t < r+2s$

These bounds are the same as in Corollary 5.2, therefore if $t \geq 2s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2t+1$, and if $t < 2s$, then $\mathcal{X}_{r,s,t}(K_{1,3}) = 2s+t+1$. □

The results presented in this subsection are summarized in Table 5.1.

## 5.3 $K_{1,n}$

The proof method used for $K_{1,3}$ (analysis of all possible constellations of its elements) is obviously not applicable in general, due to the length of the resulting proof. But the colourings used in the basic case will be very useful to find proper colourings for $K_{1,n}$ and give upper bounds for its $[r,s,t]$-chromatic number.

| Conditions | | | | | | $\mathcal{X}_{r,s,t}(K_{1,3})$ |
|---|---|---|---|---|---|---|
| $r \geq s+2t$ | $r \geq 2s+2t$ | | | | | r+1 |
| | $r < 2s+2t$ | $r \geq 2s+t$ | | | | 2s+2t+1 |
| | | $r < 2s+t$ | $r \geq 2s$ | | | r+t+1 |
| | | | $r < 2s$ | $2s > r+t$ | $2s \leq r+2t$ | r+2t+1 |
| | | | | | $2s > r+2t$ | 2s+1 |
| | | | | $2s \leq r+t$ | | 2s+t+1 |
| $(s+t \leq r < s+2t)$ $\wedge(r \geq 2t)$ | $2s+t \geq r$ | $r \geq 2s$ | | | | r+t+1 |
| | | $r < 2s$ | $2s \geq r+t$ | | | 2s+1 |
| | | | $2s < r+t$ | | | 2s+t+1 |
| | $2s+t < r$ | | | | | 2s+2t+1 |
| $(s \leq r < s+t)$ $\wedge(r \geq 2t)$ | $2s \geq r+2t$ | | | | | 2s+1 |
| | $r+t \leq 2s < r+2t$ | | | | | r+2t+1 |
| | $2s < r+t$ | | | | | 2s+t+1 |
| $(s \geq r) \wedge (s \geq 2t)$ | | | | | | 2s+1 |
| $t < r, s < 2t$ | $r \geq s$ | $2s \geq r+t$ | | | | r+2t+1 |
| | | $2s < r+t$ | | | | 2s+t+1 |
| | $r < s$ | | | | | s+2t+1 |
| $r \leq t \leq s < 2t$ | | | | | | s+2t+1 |
| $s \leq t \leq r < 2t$ | $s+t \geq r$ | | | | | s+2t+1 |
| | $s+t < r$ | $2s+t \geq r$ | | | | r+t+1 |
| | | $2s+t < r$ | | | | 2s+2t+1 |
| $r, s \leq t < r+s$ | $r \geq s$ | | | | | r+s+t+1 |
| | $r < s$ | | | | | 2s+t+1 |
| $t \geq r+s$ | $t \geq r+2s$ | | | | | r+2s+t+1 |
| | $t < r+2s$ | $t \geq 2s$ | | | | 2t+1 |
| | | $t < 2s$ | | | | 2s+t+1 |

Table 5.1: $[r, s, t]$-chromatic number of $K_{1,3}$

As observed in Section 2.1, a star is a bipartite graph, hence $\mathcal{X}(K_{1,n}) = 2$ and $\mathcal{X}'(K_{1,n}) = \Delta(K_{1,n}) = n$. From these values and Lemma 5.1 follows Corollary 5.13.

**Corollary 5.13.** *For all* $n \geq 3$

$\max\{r+1, (n-1)s+1, \min\{r+(n-1)s+t+1, (n-2)s+2t+1, \max\{r+1, 2t+1, (n-1)s+t+1\}\}\} \leq \mathcal{X}_{r,s,t}(K_{1,n}) \leq r+(n-1)s+t+1.$

**Theorem 5.14.** *If* $r \geq (n-2)s + 2t$*, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} r+1 & \text{if } r \geq (n-1)s + 2t; \\ \leq (n-1)s + 2t + 1 & \text{if } (n-1)s + t \leq r < (n-1)s + 2t; \\ \leq r + t + 1 & \text{if } (n-1)s \leq r < (n-1)s + t; \\ \leq (n-1)s + t + 1 & \text{if } r < (n-1)s \leq r + t; \\ \leq r + 2t + 1 & \text{if } r + t < (n-1)s \leq r + 2t; \\ 2s + 1 & \text{if } (n-1)s > r + 2t. \end{cases}$$

*Proof.* For $r \geq (n-2)s + 2t$, the colouring given by Figure 5.21 implies that if $r \geq (n-1)s+2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r+1$ and if $r < (n-1)s+2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s + 2t + 1$.

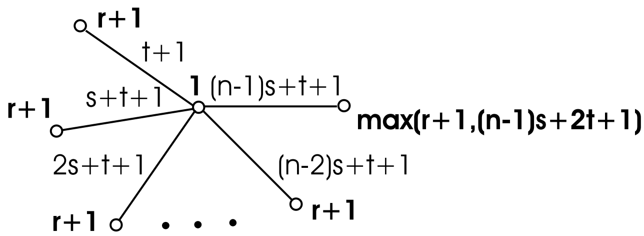



Figure 5.21: $r \geq (n-1)s + 2t$

Hence if $r \geq (n-1)s + 2t$, by Corollary 5.13 $\mathcal{X}_{r,s,t}(K_{1,n}) = r + 1$.

On the other hand, if $r < (n-1)s+t$ the colouring shown in Figure 5.22 is possible.

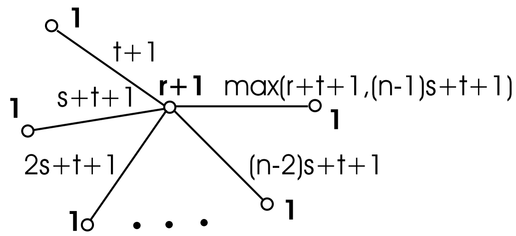



Figure 5.22: $(n-1)s \leq r < (n-1)s + t$

Hence if $(n-1)s \leq r < (n-1)s+t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r+t+1$ and if $r < (n-1)s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s + t + 1$.

But, if $r + t < (n-1)s$ a colouring with less colours can be found (see Figure 5.23).

Hence if $r + t < (n-1)s \leq r + 2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r + 2t + 1$, and if $(n-1)s > r + 2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s + 1$, hence in the last case by Corollary 5.13, $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s + 1$. □

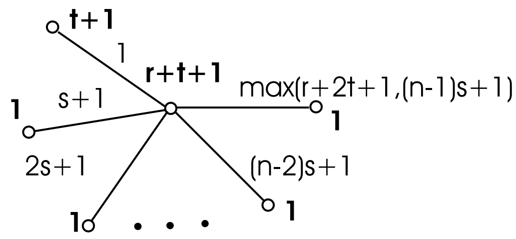



Figure 5.23: $r + t < (n-1)s$

**Theorem 5.15.** *If* $(n-2)s + t \leq r < (n-2)s + 2t$ *and* $r \geq 2t$*, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} \leq (n-1)s + 2t + 1 & \textit{if} \quad r > (n-1)s + t; \\ \leq r + t + 1 & \textit{if} \quad (n-1)s \leq r \leq (n-1)s + t; \\ \leq 2s + t + 1 & \textit{if} \quad r < (n-1)s \leq r + t; \\ 2s + 1 & \textit{if} \quad (n-1)s > r + t. \end{cases}$$

*Proof.* *a)* If $\mathbf{r} > (\mathbf{n}-\mathbf{1})\mathbf{s} + \mathbf{t}$, then from the colouring shown in Figure 5.24, it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s + 2t + 1$.

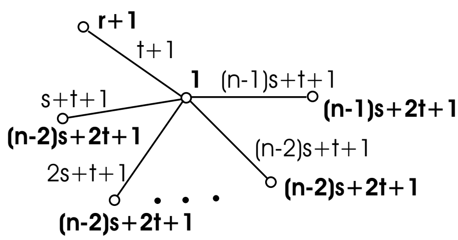



Figure 5.24: $(n-1)s + t \leq r < (n-2)s + 2t$, $r \geq 2t$

*b)* If $(\mathbf{n}-\mathbf{1})\mathbf{s} \leq \mathbf{r} \leq (\mathbf{n}-\mathbf{1})\mathbf{s} + \mathbf{t}$, the colouring given in Figure 5.25 shows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r + t + 1$.

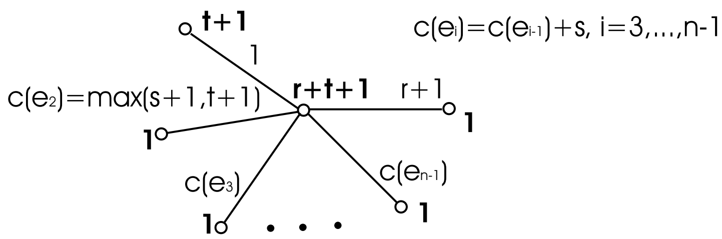



Figure 5.25: $(n-2)s + t \leq r < (n-2)s + 2t$, $r \geq 2t$ and $(n-1)s \leq r \leq (n-1)s + t$

*c)* If $\mathbf{r} < (\mathbf{n}-\mathbf{1})\mathbf{s} < \mathbf{r}+\mathbf{t}$ from the colouring shown by Figure 5.26, it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+t+1$.

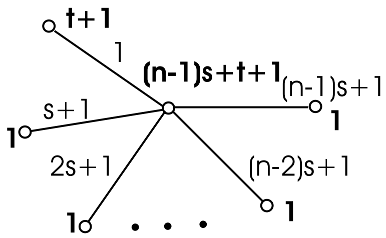



Figure 5.26: $(n-2)s+t \leq r < (n-2)s+2t$ and $2t \leq r < (n-1)s < r+t$

And if $(n-1)s > r+t$, then the colouring shown in Figure 5.27 is possible.

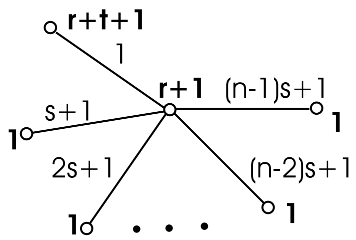



Figure 5.27: $(n-2)s+t \leq r < (n-2)s+2t$, $r \geq 2t$ and $(n-1)s > r+t$

Hence, $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+1$ and by Corollary 5.13 it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+1$. $\square$

**Theorem 5.16.** *If $(n-2)s \leq r < (n-2)s+t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} (n-1)s+1 & \text{if} \quad (n-1)s \geq r+2t; \\ \leq r+2t+1 & \text{if} \quad r+t \leq (n-1)s < r+2t; \\ \leq (n-1)s+t+1 & \text{if} \quad (n-1)s < r+t \text{ and } s \geq t; \\ \leq (n-2)s+2t+1 & \text{if} \quad (n-1)s < r+t \text{ and } s < t. \end{cases}$$

*Proof.* For $(n-2)s \leq r < (n-2)s+t$ and $r \geq 2t$ the colouring given in Figure 5.28 proves that if $(n-1)s \geq r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+1$ and if $r+t \leq (n-1)s < r+2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r+2t+1$. Then by Corollary 5.13, in the first case $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+1$.

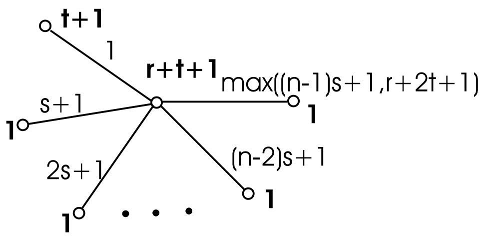



Figure 5.28: $(n-2)s \leq r < (n-2)s+t$, $r \geq 2t$ and $(n-1)s \geq r+2t$

If $(n-1)s < r+t$, than less colours can be used. But in order to be able to use the colouring given by Figure 5.29 (similar to the colouring for $K_{1,3}$ in Figure 5.10), $s$ should be greater than $t$, which is not always the case (observe that for $K_{1,3}$ this fact followed from $s+t \geq r > 2t$).

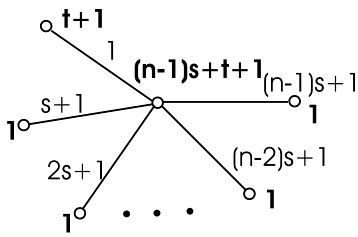



Figure 5.29: $(n-2)s \leq r < (n-2)s+t$, $r \geq 2t$, $(n-1)s < r+t$ and $s \geq t$

Then if $(n-1)s < r+t$ and $s \geq t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+t+1$.

On the other hand, if $s < t$, from the colouring shown in Figure 5.30, it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-2)s+2t+1$.

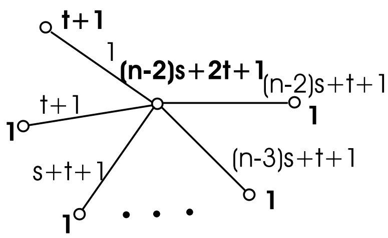



Figure 5.30: $(n-2)s \leq r < (n-2)s+t$, $r \geq 2t$, $(n-1)s < r+t$ and $s < t$

□

In order to cover all possible situations and due to the role of $n$, a new situation (in relation with the situations studied for $K_{1,3}$) has to be considered and is presented in Theorem 5.17.

**Theorem 5.17.** *If $r < (n-2)s + t$ and $r \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} (n-1)s+1 & \text{if} \quad s \geq 2t; \\ \leq (n-1)s+t+1 & \text{if} \quad t \leq s < 2t; \\ \leq (n-2)s+2t+1 & s < t. \end{cases}$$

*Proof.* *a)* If $\mathbf{s} \geq \mathbf{2t}$, then the colouring given by Figure 5.31 implies that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+1$.

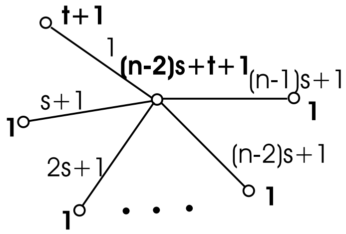



Figure 5.31: $r < (n-2)s + t$, $r \geq 2t$ and $s \geq 2t$

Then by Corollary 5.13, $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+1$.

*b)* If $\mathbf{s} < \mathbf{2t}$ from the colouring shown in Figure 5.32, it follows that if $t \leq s < 2t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+t+1$ and if $s < t$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-2)s+2t+1$.

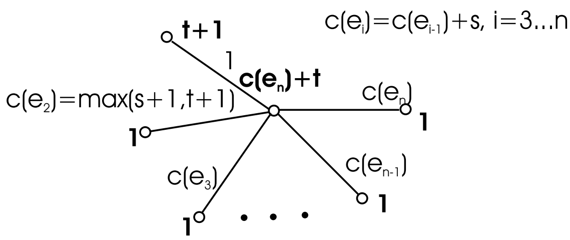



Figure 5.32: $r < (n-2)s + t$, $r \geq 2t$ and $s < 2t$

□

**Theorem 5.18.** *If $s \geq r$ and $s \geq 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+1$$

*Proof.* From the colouring shown in Figure 5.33 it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+1$.

Then by Corollary 5.13, $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+1$. □

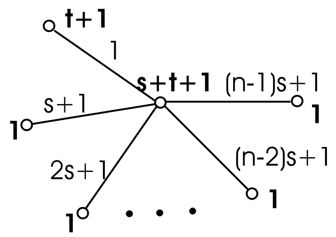



Figure 5.33: $s \geq r$ and $s \geq 2t$

**Theorem 5.19.** *If $t < r, s < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} \leq r + 2t + 1 & \textit{if} \quad r \geq (n-2)s \textit{ and } (n-1)s \geq r + t \ ; \\ \leq (n-1)s + t + 1 & \textit{if} \quad r \geq (n-2)s \textit{ and } (n-1)s < r + t; \\ (n-2)s + 2t + 1 & \textit{otherwise.} \end{cases}$$

*Proof. a)* If $\mathbf{r} \geq (\mathbf{n}-\mathbf{2})\mathbf{s}$ from the colouring shown by Figure 5.34, it follows that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r + 2t + 1$.

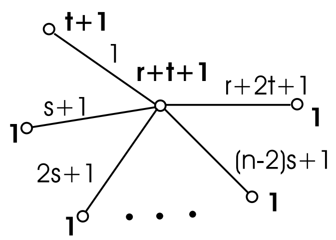



Figure 5.34: $t < r, s < 2t$ and $r \geq (n-2)s$

*b)* If in addition $(\mathbf{n}-\mathbf{1})\mathbf{s} < \mathbf{r}+\mathbf{t}$, then the colouring given by Figure 5.35 shows that at most $2s + t + 1$ colours are needed.

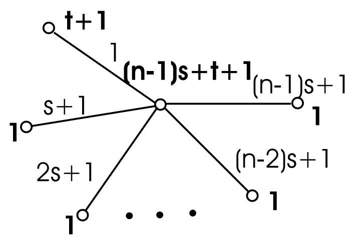



Figure 5.35: $t < r, s < 2t$ and $r + s \leq (n-1)s < r + t$

*c)* If $\mathbf{r} < (\mathbf{n}-\mathbf{2})\mathbf{s}$, the colouring given by Figure 5.36 demostrates that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-2)s+2t+1$ which coincides with the lower bound given by Corollary 5.13. Hence $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-2)s + 2t + 1$. ☐

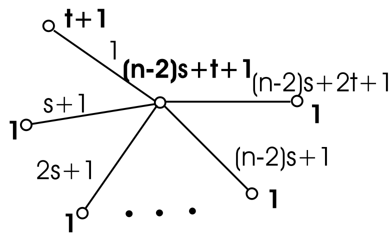



Figure 5.36: $t < r, s < 2t$ and $r < (n-2)s$

**Theorem 5.20.** *If $r \leq t \leq s < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,3}) = (n-2)s + 2t + 1.$$

*Proof.* The colouring given by Figure 5.37 gives an upper bound that coincides with the lower bound given by Corollary 5.13. Hence, $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-2)s + 2t + 1$.

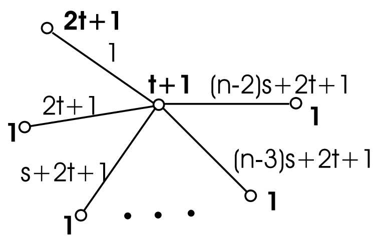



Figure 5.37: $r \leq t \leq s < 2t$

□

**Theorem 5.21.** *If $s \leq t \leq r < 2t$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} \leq (n-2)s + 2t + 1 & \textit{if} \quad (n-2)s + t \geq r; \\ \leq r + t + 1 & \textit{if} \quad (n-2)s + t < r \leq 2s + t; \\ \leq (n-1)s + 2t + 1 & \textit{otherwise.} \end{cases}$$

*Proof.* The colouring given by Figure 5.38 proves that if $(n-2)s + t \geq r$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-2)s + 2t + 1$ and if $(n-2)s + t < r$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r + t + 1$.

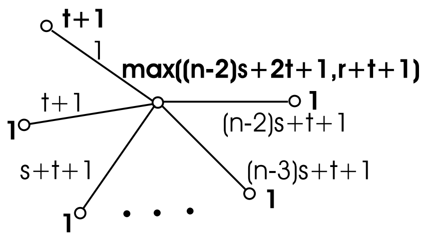



Figure 5.38: $s \leq t \leq r < 2t$

But if $(n-1)s + t \leq r$, then a colouring with less colours is possible (see Figure 5.39) and it follows that $\mathcal{X}_{r,s,t}(K_{1,3}) \leq 2s + 2t + 1$. $\square$

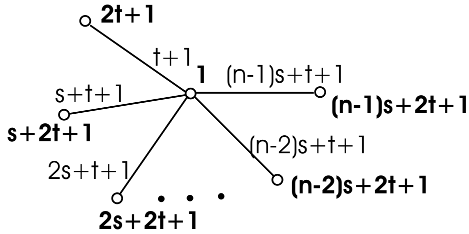



Figure 5.39: $(n-1)s + t \leq r < 2t$

**Theorem 5.22.** *If* $r, s \leq t < r + s$, *then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} \leq r + (n-2)s + t + 1 & \textit{if} \quad r \geq s; \\ (n-1)s + t + 1 & \textit{otherwise.} \end{cases}$$

*Proof.* From the colouring given by Figure 5.40, it follows that if $r \geq s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r + (n-1)s + t + 1$ and if $r < s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s + t + 1$.

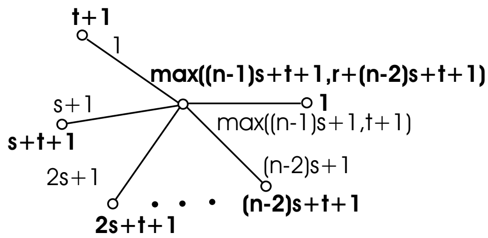



Figure 5.40: $r, s \leq t < r + s$

Furthermore, in the second case the upper bound coincides with the lower bound given by Corollary 5.13, hence $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s + t + 1$ if $r < s$. $\square$

**Theorem 5.23.** *If $t \geq r+s$, then*

$$\mathcal{X}_{r,s,t}(K_{1,n}) = \begin{cases} r+(n-1)s+t+1 & \text{if} \quad t \geq r+(n-1)s; \\ 2t+1 & \text{if} \quad (n-1)s \leq t < r+(n-1)s; \\ (n-1)s+t+1 & \text{otherwise.} \end{cases}$$

*Proof.* *a)* If $\mathbf{t} \geq \mathbf{r}+(\mathbf{n}-\mathbf{1})\mathbf{s}$ the colouring given by Figure 5.41 implies that $\mathcal{X}_{r,s,t}(K_{1,n}) \leq r+(n-1)s+t+1$. This is an upper bound coincides with the lower bound given by Corollary 5.13.

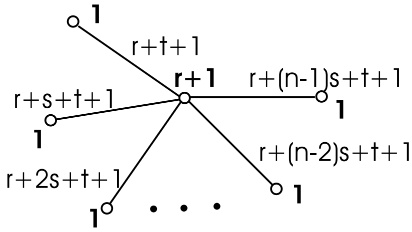



Figure 5.41: $t \geq r+(n-1)s$

Hence, $\mathcal{X}_{r,s,t}(K_{1,n}) = r+(n-1)s+t+1$.

*b)* If $\mathbf{t} < \mathbf{r}+(\mathbf{n}-\mathbf{1})\mathbf{s}$ from the colouring shown by Figure 5.42, it follows that if $t \geq (n-1)s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq 2t+1$ and if $t < (n-1)s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) \leq (n-1)s+t+1$. These bounds are the same as given by Corollary 5.13, therefore

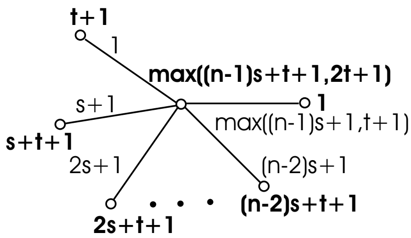



Figure 5.42: $r+s \leq t < r+2s$

if $t \geq (n-1)s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) = 2t+1$, and if $t < (n-1)s$, then $\mathcal{X}_{r,s,t}(K_{1,n}) = (n-1)s+t+1$. $\square$

The results presented in this section are summarized in Table 5.2.

| Conditions | | | | $\mathcal{X}_{r,s,t}(K_{1,n})$ |
|---|---|---|---|---|
| $r \geq (n-2)s + 2t$ | $r \geq (n-1)s + 2t$ | | | $r + 1$ |
| | $r < (n-1)s + 2t$ | $r \geq (n-1)s + t$ | | $\leq (n-1)s + 2t + 1$ |
| | | $r < (n-1)s + t$ | $r \geq (n-1)s$ | $\leq r + t + 1$ |
| | | | $r < (n-1)s$ | $\leq (n-1)s + t + 1$ |
| $((n-2)s + t \leq r$ $< (n-2)s + 2t)$ $\wedge(r \geq 2t)$ | $(n-1)s + t \geq r$ | $r \geq (n-1)s$ | | $\leq r + t + 1$ |
| | | $r < (n-1)s$ | $(n-1)s \geq r + t$ | $(n-1)s + 1$ |
| | | | $(n-1)s < r + t$ | $\leq (n-1)s + t + 1$ |
| | $(n-1)s + t < r$ | | | $\leq (n-1)s + 2t + 1$ |
| $((n-2)s \leq r <$ $(n-2)s + t)$ $\wedge(r \geq 2t)$ | $(n-1)s \geq r + 2t$ | | | $(n-1)s + 1$ |
| | $r + t \leq (n-1)s < r + 2t$ | | | $\leq r + 2t + 1$ |
| | $(n-1)s < r + t$ | $s \geq t$ | | $\leq (n-1)s + t + 1$ |
| | | $s < t$ | | $\leq (n-2)s + 2t + 1$ |
| $(r < (n-2)s)$ $\wedge(r \geq 2t)$ | $s \geq 2t$ | | | $(n-1)s + 1$ |
| | $s < 2t$ | $s \geq t$ | | $\leq (n-1)s + t + 1$ |
| | | $s < t$ | | $\leq (n-2)s + 2t + 1$ |
| $(s \geq r) \wedge (s \geq 2t)$ | | | | $(n-1)s + 1$ |
| $t < r, s < 2t$ | $r \geq (n-2)s$ | $(n-1)s \geq r + t$ | | $\leq r + 2t + 1$ |
| | | $(n-1)s < r + t$ | | $\leq (n-1)s + t + 1$ |
| | $r < (n-2)s$ | | | $(n-2)s + 2t + 1$ |
| $r \leq t \leq s < 2t$ | | | | $(n-2)s + 2t + 1$ |
| $s \leq t \leq r < 2t$ | $(n-2)s + t \geq r$ | | | $\leq (n-2)s + 2t + 1$ |
| | $(n-2)s + t < r$ | $(n-1)s + t \geq r$ | | $\leq r + t + 1$ |
| | | $(n-1)s + t < r$ | | $\leq (n-1)s + 2t + 1$ |
| $r, s \leq t < r + s$ | $r \geq s$ | | | $\leq r + (n-2)s + t + 1$ |
| | $r < s$ | | | $(n-1)s + t + 1$ |
| $t \geq r + s$ | $t \geq r + (n-1)s$ | | | $r + (n-1)s + t + 1$ |
| | $t < r + (n-1)s$ | $t \geq (n-1)s$ | | $2t + 1$ |
| | | $t < (n-1)s$ | | $(n-1)s + t + 1$ |

Table 5.2: $[r, s, t]$-chromatic number of $K_{1,n}$

# Chapter 6

# Other Graph Classes

In this chapter some results for other classes of graphs are presented.

## 6.1 Bipartite Graphs

As noted before, paths, even cycles and stars are examples of bipartite graphs. Since the chromatic number and chromatic index are know for them (Lemma 2.6 and Theorem 2.11), a "good" $[r, s, t]$-colouring can be found in general if they are coloured with a certain rule.

**Theorem 6.1.** *If $G$ is a bipartite graph, a colouring using $k$ colours can be easily found in such a way that*

$$k - \mathcal{X}_{r,s,t}(G) \leq 2t.$$

*Proof.* Since $\mathcal{X}(G) = 2$ and $\mathcal{X}'(G) = \Delta$ (where $\Delta := \Delta(G)$) for all bipartite graph, $G$, from Lemma 2.25 it follows that

$$\max\{r+1, s(\Delta-1)+1, \min\{r+s(\Delta-1)+t+1, s(\Delta-2)+2t+1, \max\{r+1, 2t+1, s(\Delta-1)+t+1\}\}\} \leq \mathcal{X}_{r,s,t}(G) \leq r+s(\Delta-1)+t+1.$$

Then, if $r > s(\Delta-1)+2t$, the vertices can be coloured with the two colours 1 and $r+1$ and the edges with the $\Delta$ colours in between: $t+1, s+t+1, \ldots, s(\Delta-1)+t+1$. Hence just $r+1$ colours have to be used and so, because of the previous bounds, $\mathcal{X}_{r,s,t}(G) = r+1$.

On the other hand, if $r \leq s(\Delta-1)+2t$, the edges can be coloured with the colours $t+1, s+t+1, \ldots, s(\Delta-1)+t+1$ and the vertices can receive the colours $1, s(\Delta-1)+2t+1$. Hence, a colouring using $s(\Delta-1)+2t$ colours has been found and, since $\mathcal{X}_{r,s,t}(G) \geq s(\Delta-1)$, it differs at most on $2t$ from an optimal colouring. $\square$

Observe that then, if $t = 0$, the colourings described above are optimal. On the other hand, it is obvious since then vertex-colouring and edge-colouring the graph would be independent processes.

## 6.2 Complete Graphs

From Lemma 2.22, it follows that $\mathcal{X}_{r,s,t}(K_{\omega(G)}) \leq \mathcal{X}_{r,s,t}(G) \leq \mathcal{X}_{r,s,t}(K_{|V(G)|})$, where $|V(G)|$ is the order of $G$. Therefore, the study of the $[r,s,t]$-chromatic number of complete graphs is of great interest.

Complete graphs were already investigated by Kemnitz and Marangio [11], who determined the exact value of $\mathcal{X}_{r,s,t}(K_n)$ if $\min\{r,s,t\} = 0$ in almost any case. Moreover, if $\min\{r,s,t\} \geq 1$ the following Theorem was given.

**Theorem 6.2.** *If* $\min\{r,s,t\} \geq 1$, $n \geq 3$ *and* $\Delta = \Delta(K_n)$, *then*

*a)*$\mathcal{X}_{r,r,r}(K_n) = r\Delta + 1$ *if* $n$ *odd;*
*b)*$\mathcal{X}_{r,r,r}(K_n) = r(\Delta + 1) + 1$ *if* $n$ *even;*
*c)*$\mathcal{X}_{r,s,t}(K_{2n+1}) = r\Delta + 1,$ *if* $1 \leq s \leq r$ *and* $1 \leq t \leq r$;
*d)*$\mathcal{X}_{r,s,t}(K_{2n+1}) = s\Delta + 1$ *if* $1 \leq r \leq s$ *and* $1 \leq t \leq s$;
*e)*$\mathcal{X}_{r,s,t}(K_{2n}) = r\Delta + 1$ *if* $r \geq 2$, $1 \leq s \leq r$ *and* $1 \leq t \leq \lfloor r/2 \rfloor$;
*f)*$\mathcal{X}_{1,s,1}(K_{2n}) = s(\Delta - 1) + 1$ *if* $s \geq 3$;
*g)*$\mathcal{X}_{1,2,1}(K_{2n}) = 2\Delta$;
*h)*$\mathcal{X}_{r,s,t}(K_n) = r\Delta + s(\Delta - 1) + t + 1$ *if* $t > (r+s)\Delta$ *and* $r \geq s$.

Then, the smallest cases with $\min\{r,s,t\} \geq 1$ which are not covered by Theorem 6.2 are $\mathcal{X}_{2,3,1}(K_{2n})$ and $\mathcal{X}_{1,1,2}(K_n)$. The first value is determined in the next Theorem.

**Theorem 6.3.** $\mathcal{X}_{2,3,1}(K_{2n}) = 6n - 3$, *for all* $n$.

*Proof.* Kemnitz and Marangio [11] gave a first approximation as follows: $6n - 4 \leq \mathcal{X}_{2,3,1}(K_{2n}) \leq 6n - 3$.
Suppose $\mathcal{X}_{2,3,1}(K_{2n}) = 6n - 4$. Then since $\mathcal{X}(K_{2n}) = \Delta + 1$ (where $\Delta = 2n - 1$) and $\mathcal{X}'(K_{2n}) = \Delta$, at least this number of different colours should be used for the vertices and the edges, respectively.
But no more than $\Delta$ different colours can be used for the edges, since otherwise $s(\Delta + 1 - 1) + 1 = 6n - 3 + 1 = 6n - 2 > 6n - 4$ colours would be at least needed. Then, denote this $\Delta$ colours as $c_1^e < c_2^e < \cdots < c_\Delta^e$ (analogously the colours for the vertices will be $c_1^v < c_2^v < \cdots < c_{\Delta+1}^v < c_{\Delta+2}^v < \dots$).
The possible situations are the following:

Case 1: There exist $c_i^v, c_j^v$ such that $c_i^v, c_j^v < c_1^e$. Then $\mathcal{X}_{2,3,1}(K_{2n}) \geq s(\Delta - 1) + 1 + t + r = 6n - 2$, which is a contradiction.

Case 2: There exist $c_i^v, c_j^v$ such that $c_i^v < c_1^e$ and $c_j^v > c_\Delta^e$. Then $\mathcal{X}_{2,3,1}(K_{2n}) \geq t + 1 + s(\Delta - 1) + t = 6n - 3$, which is not possible.

Case 3: There exists just one $c_i^v$ such that $c_i^v > c_\Delta^e$ (or $c_i^v < c_1^e$). Then, there must exist $c_p^v, c_t^v, c_m^e$ such that $c_m^e < c_p^v < c_t^v < c_{m+1}^e$. Then $\mathcal{X}_{2,3,1}(k_{2n}) \geq s(m-1) + 1 + t + r + t + s(\Delta - m - 1) + t = 6n - 3$, a contradiction.

Case 4: The only possible remaining situations are those where between two edge-colours are at least three vertex-colours or where at least more than one vertex-colour

is between two pairs of edge-colours. In detail:
Case 4(1): If $c_1^e < \cdots < c_m^e < c_i^v < c_j^v < c_p^v < c_{m+1}^e < \cdots < c_\Delta^e$, for some $m, i, j, p$. Then $\mathcal{X}_{2,3,1}(K_{2n}) \geq s(m-1)+1+t+2r+t+s(\Delta-m-1) = 6n-2$, a contradiction.
Case 4(2): If $c_1^e < \cdots < c_m^e < c_i^v < c_j^v < c_{m-1}^e < \cdots < c_k^e < c_p^v < c_q^v < c_{k+1}^e < \cdots < c_\Delta^e$, for some $m, i, j, k, p, q$. Then $\mathcal{X}_{2,3,1}(K_{2n}) \geq s(m-1)+1+t+r+t+s(k-m-1)+t+r+t+s(\Delta-k-1) = 6n-3$, which is a contradiction.

Hence, $6n-4$ colours are not enough for a $[2,3,1]$-colouring of $K_{2n}$. So $\mathcal{X}_{2,3,1}(K_{2n}) = 6n-3$, for all $n$. $\square$

# Chapter 7

# Summary

$[r, s, t]$-colourings of graphs have been introduced as an extension of the classical graph colourings: vertex-colouring, edge-colouring and total-colouring. As mentioned in Section 2.1, these colourings have been studied for a long time. So the exact value of the chromatic number, the edge-chromatic number and the total-chromatic number has been determined for many classes of graphs. In Section 2.1 the results for paths, cycles, stars, bipartite graphs and complete graphs are presented and are now summarized in Table 7.1.

| Class of graphs | $\mathcal{X}(G)$ | $\mathcal{X}'(G)$ | $\mathcal{X}_T(G)$ |
|---|---|---|---|
| Paths ($P_n$) | 2 | 2 | 3 |
| Cycles ($C_n$) | 2 if n even<br>3 if n odd | 2 if $n$ even<br>3 if $n$ odd | 3 if $n \equiv 0$ $mod$ 3<br>4 if $n \neq 0$ $mod$ 3 |
| Stars ($K_{1,n}$) | 2 | $n$ | $n+1$ |
| Bipartite | 2 | $\Delta(G)$ | $\Delta(K_{n,m})+1$ if complete bip. and $n \neq m$<br>$\Delta(K_{n,m})$ if complete bip. and $n = m$ |
| Complete ($K_n$) | $n$ if $n$ even<br>$n$ if $n$ odd | $n-1$ if $n$ even<br>$n$ if $n$ odd | $n-1$ if $n$ even<br>$n$ if $n$ odd |

Table 7.1: Classical Colourings

However, the chromatic number, edge-chromatic number and total-chromatic number have not been determined for every class of graphs, since this is a hard task. This shows the difficulty of any generalization of vertex-, edge- or total-colouring, which is the case of the $[r, s, t]$-colouring.

In Chapter 1, applications of these colourings for solving different scheduling problem have been shown. There, we noted that considering some natural constraints (for example relating waiting, resting or preparing times) a new notion of colouring should be introduced.

This was realized by Hackmann, Kemnitz and Marangio [11] by defining the $[r, s, t]$-colouring as follows: Given non-negative integers $r$, $s$ and $t$, an $[r, s, t]$-colouring of a graph $G = (V(G), E(G))$ is a mapping $c$ from $V \cup E(G)$ to the colour set $\{1, 2, ..., k\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices $v_i, v_j$, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges $e_i, e_j$, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices $v_i$ and edges $e_j$, respectively. The $[r, s, t]$-chromatic number $\mathcal{X}_{r,s,t}(G)$ of $G$ is defined as the minimum $k$ such that $G$ admits an $[r, s, t]$-colouring.

This definition (Definition 2.21) and some hereditary properties (Lemmas 2.22 and 2.23) can be found in Section 2.2. There is also given a first improvement of the general bounds by Kemnitz and Marangio, which says that

$$\max\{r(\mathcal{X}(G)-1)+1, s(\mathcal{X}'(G)-1)+1, t+\Delta(G)\} \leq \mathcal{X}_{r,s,t}(G) \leq r(\mathcal{X}(G)-1)+s(\mathcal{X}'(G)-1)+t+1.$$

Its sharpness was also shown, although the third term of the lower bounds is sharp only in a very specific case, which made us believe that it could be still improved. In fact it was done later in Chapter 5.

As a first step, the simplest class of graphs was taken. In Chapter 3 the $[r, s, t]$-colouring for paths was studied, using the previous bounds in some situations and considering all possible constellations of the colours of a subpath of order 3 in other cases. Then the $[r, s, t]$-chromatic number for paths could be determined in all possible situations, as listed in Table 3.1.

Furthermore, the study of the $[r, s, t]$-colouring for paths leads to useful applications. Because of Lemma 2.22, the $[r, s, t]$-chromatic number of a path can be used as a lower bound of the correspondent value for any graph containing it as a subgraph. This fact was used in Chapter 4 to determine the $[r, s, t]$-chromatic number for cycles.

Cycles of even and odd order have different chromatic number, edge-chromatic number and total-chromatic number, which affects for instance the general lower bounds given by Lemma 2.25. Due to this fact, these two cases were treated separately in Sections 4.1 and 4.2, respectively.
On the other hand, for cycles whose order is not a multiple of 3, $\mathcal{X}_T(C_n) = 4$. This was used to study all possible constellations of colours of the elements of the cycle and the lower bounds that they imply, which reduce the possible situations to be studied. This fact was very useful to present a complete listing of the $[r, s, t]$-chromatic number for cycles, as shown in Tables 4.1 and 4.2.

According to Lemma 2.22, $\mathcal{X}_{r,s,t}(K_{1,\Delta(G)}) \leq \mathcal{X}_{r,s,t}(G)$. Hence, the study of the $[r, s, t]$-chromatic number for stars is specially interesting, since it provides a lower bound for any graph $G$ in relation with its maximum degree.
This task is presented in Chapter 5. At first, a second improvement of the general bounds by Kemnitz and Marangio was given as follows:

$$\max\{r + 1, 2s + 1, \min\{r + 2s + t + 1, s + 2t + 1, \max\{r + 1, 2t + 1, 2s + t + 1\}\}\} \leq$$
$$\leq \mathcal{X}_{r,s,t}(K_{1,3}) \leq r + 2s + t + 1.$$

These bounds are often sharp for stars.
However, there were some remaining cases, for which a similar strategy as the one used for cycles should be applied. Therefore, all possible constellations of the colours

of its elements with at least one monotone sequence of length four were listed, since $\mathcal{X}_T(K_{1,3}) = 4$. In this way the situations to be studied were reduced, as expected. Using this strategy the $[r, s, t]$-chromatic number for $K_{1,3}$ could be determined. The results are presented in Section 5.2 and summarized in Table 5.1.

The colourings used for $K_{1,3}$ give a schema of how the colourings for the general case should be. Applying these colourings and the improved general bounds, the $[r, s, t]$-chromatic number for $K_{1,n}$ could be determined in some cases. In the remaining cases, bounds were given. All these results are presented in Section 5.3 and summarized in Table 5.2.

Finally in Chapter 6 some other classes of graphs are studied. In particular, it is proved that for bipartite graphs a good approximation to the optimal $[r, s, t]$-colouring can be always easily found. And for complete graphs, which were already studied by Kemnitz and Marangio [11], the $[r, s, t]$-chromatic number for a special case is given.

# Glossary

$[r, s, t]$-chromatic number: see Definition 2.21

$[r, s, t]$-colouring: see Definition 2.21

$\Delta := \Delta(G)$: Maximum vertex degree of $G$

$\mu(G)$: Maximum multiplicity of edges in $G$

$\omega(G)$: Clique number of $G$

$C_n$: Cycle of order $n$

$E(G)$: Set of edges of $G$

$G^*$: Dual graph of $G$

$H \subseteq G$: $H$ is a subgraph of $G$

$K_{1,n}$: Star of order $n + 1$ (with $n$ leaves)

$K_n$: Complete graph of order $n$

$L(G)$: Line graph of $G$

$N_n$: Empty graph

$P_n$: Path of order $n$

$V(G)$: Set of vertices of $G$

$\mathcal{X}'(G)$: Edge-chromatic number of $G$

$\mathcal{X}(G)$: Chromatic number of $G$

$\mathcal{X}_{r,s,t}(G)$: $[r, s, t]$-chromatic number

$\mathcal{X}_T(G)$: Total-chromatic number of $G$

Acyclic graph: Graph without cycles

Bipartite graph: Graph, whose vertex set is the union of two disjoint independent sets

Chromatic number of $G$: see Definition 2.1

Clique number of $G$: see page 7

Complement graph of $G$, $\overline{G}$: Simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv$ is not an element of $E(G)$

Complete graph: Simple graph, whose vertices are pairwise adjacent

Components of $G$: Maximal connected subgraphs in $G$

Connected graph: A graph where each pair of vertices belongs to a path

Cycle: Graph with equal number of vertices and edges, $n$, whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle

Dual graph: see page 6

Edge-chromatic number (or chromatic index): see Definition 2.7

Edge-colouring: see Definition 2.7

Empty graph: Graph with no edges

Independent set: Set of pairwise nonadjacent vertices

Line graph: see Definition 2.8

Matching: Set of non-loop edges with no shared endpoints

Maximum (vertex) degree of $G$: Maximum number of incident edges to any vertex of $G$

Multiple edges: Edges having the same end points

NP-complete problem: Problem for which no polynomial-time algorithm solving it is known

Order of a graph: Number of vertices

Path: Simple graph whose $n$ vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list

Perfect matching: Matching that saturates every vertex

Planar graph: Graph that can be drawn in the plane without crossing edges

Regular graph: Graph $G$, such that $\Delta(G) = \delta(G)$, where $\delta(G)$ is the minimum degree of $G$ and is defined analogously to $\Delta(G)$

Simple graph: Graph not having loops or multiple edges

Star: Connected acyclic graph consisting of one vertex adjacent to all the others

Subgraph: $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of end points to edges in $H$ is the same as in $G$

Total-chromatic number: see Definition 2.14

Total-colouring: see Definition 2.14

Vertex-colouring: see Definition 2.1

# List of Tables

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