



Graph Polynomials and Their Representations

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submitted by **Dipl.-Math. (FH) Martin Trinks**

born on the 15.05.1984 in Mittweida

Assessor: Prof. Ingo Schiermeyer und Prof. Peter Tittmann

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Abstract

Graph polynomials are polynomials associated to graphs that encode the number of subgraphs with given properties. We list different frameworks used to define graph polynomials in the literature. We present the edge elimination polynomial and introduce several graph polynomials equivalent to it. Thereby, we connect a recursive definition to the counting of colorings and to the counting of (spanning) subgraphs. Furthermore, we define a graph polynomial that not only generalizes the mentioned, but also many of the well-known graph polynomials, including the Potts model, the matching polynomial, the trivariate chromatic polynomial and the subgraph component polynomial. We prove a recurrence relation for this graph polynomial using edge and vertex operation. The definitions and statements are given in such a way that most of them are also valid in the case of hypergraphs.

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Chapter 1

Introduction

In graph theory, as in discrete mathematics in general, not only the existence, but also the counting of objects with some given properties, is of main interest. To count and to encode the number of structures with given properties, generating functions, formally written as polynomials, are widely used. With respect to graphs, we speak about graph polynomials that count the number of subgraphs with given properties.

A graph can nowadays be easily described as the abstraction of a network. It consists of a set of vertices and a set of edges, where each edge connects at most two vertices with each other. A graph polynomial is a polynomial associated to a graph, such that the same polynomial is assigned to graphs arising from a relabeling of the vertices.

While some graph polynomials, for instance the characteristic polynomial, the chromatic polynomial, the matching polynomial and the Tutte polynomial, are already studied intensively and also their relations are well known, this does not hold for graph polynomials in general. In fact, as more and more specific graph structures, and consequently the corresponding subgraphs, have been analyzed, for many of these a generating function and thereby a graph polynomial has been defined. Hence, there is a multitude of graph polynomials — called “the zoo of graph polynomials” following a suggestion of Zaslavsky [99, footnote on page 1] — whose similarities and differences, and hence whose relations, are not yet clarified.

The main aim of this dissertation is to give a substantial contribution to the long-term goal of establishing a “general theory of graph polynomials”, a term used by Makowsky in the title of [100]. It is clear, that this will not be possible in an one-to-one-meaning, as the nature of graph polynomials differs extremely depending on the context in which these are defined.

We have both perspectives on graph polynomials, a very general one by observing in which frameworks graph polynomials can be defined, and a very specific one exploring a specific graph polynomial, the edge elimination polynomial. We bring both perspectives together by introducing several graph polynomials

which are equivalent to the edge elimination polynomial, that means these can be calculated (for a given graph) from the edge elimination polynomial (of this graph) and vice versa, but are defined in different frameworks. By using an appropriate graph polynomial we can show some properties valid for all these equivalent graph polynomials, which may be much harder to prove starting from another definition. These results provide some evidence that it makes sense to consider equivalent graph polynomials.

Related to this is the topic of recurrence relations for graph polynomials. These either can be stated for given graph polynomials, or can be used to define some. While in the first case, the problem to find (and prove) a recurrence relation, in the second case, to find a combinatorial interpretation of the specified graph polynomial may be challenging. Again, we give some general results on recurrence relations and some specific results on single graph polynomials.

Another focus lies on the definition of (slightly more general) graph polynomials unifying several of the major graph polynomials. Regarding this, we define the generalized subgraph counting polynomial. This graph polynomial generalizes two classes of graph polynomials, those satisfying a recurrence relation with respect to some edge operations and those satisfying a recurrence relation with respect to some vertex operations. We prove that the generalized subgraph counting polynomial itself also obeys a recurrence relation.

While we have mentioned only graphs until now, many results are also valid for the more general case of hypergraphs, where in a hypergraph each edge may be connect an arbitrary number of vertices.

1.1 “Graph Polynomials and Their Representations”

The title of this dissertation is chosen to include two possible meanings of the term “representation” in connection with graph polynomials.

Mainly, by “representations for graph polynomials” we mean the frameworks (ways, formalisms, concepts) used to define graph polynomials. Thus, we are talking about an “edge subset representation” and a “coloring representation” if a graph polynomial is defined as a sum over edge subsets and as a sum over colorings, respectively. As the name suggests, the chromatic polynomial is originally defined in terms of colorings and therefore by a coloring representation.

Furthermore, it seems to be possible to expand this meaning and to refer to a graph polynomial equivalent to another graph polynomial, but defined in another framework (representation), as a “representation” of the given graph polynomial. With this meaning, a representation of the chromatic polynomial is for example the adjoint polynomial, which is defined as a sum over partitions and can be derived from the chromatic polynomial by replacing the falling factorials in an appropriate formulation by powers, and vice versa.

Another frequently used term is an “expansion” of a graph polynomial, which denotes an equivalent graph polynomial (defined in another framework) yielding

exactly the same graph polynomial. Many expansions of the chromatic polynomial are known, including an “edge subset expansion” stating the chromatic polynomial as a sum over edge subsets.

While the term “representation” is rarely used in the literature and not necessarily in the same meaning, the term “expansion” with respect to graph polynomials is widely used with the same meaning. As an example, in the article “Expansions of the chromatic polynomial” Biggs [14] states several different ways the chromatic polynomial can be defined. The first usage in this direction seems to be Whitney’s “A logical expansion in mathematics” [152], where the edge subset expansion and the broken-cycle expansion of the chromatic polynomial are given (not using these terms explicitly). The term is often used as “subset expansion”, which half the times denotes what we denote as “edge subset expansion”, but in the other cases refers to a sum or product over some other ground set than the edge set.

A related term is “type analog”, which is used by Sarmiento [122] to express that (two) graph polynomials are stated using similar formalisms: “That is, both polynomials ‘look similar’ in the sense that” replacing some term by another, one graph polynomial is transformed into another.

1.2 Literature, Own Contributions and Publications

The point of origin of the present research was the definition of the edge elimination polynomial in connection with the search for graph polynomials satisfying some recurrence relations and relations between graph polynomials following from such recurrence relations. Namely we want to mention the following literature:

- “From a zoo to a zoology: Towards a general theory of graph polynomials” [100],
- “A most general edge elimination polynomial” and “An extension of the bivariate chromatic polynomial”, both introducing the edge elimination polynomial [4; 5],
- some surveys on graph polynomials [54; 55; 108; 113].

My own contributions are in particular the definition of the covered components polynomial, the subgraph counting polynomial, the trivariate chromatic polynomial (all equivalent to the edge elimination polynomial) and the generalized subgraph counting polynomial, together with the statement concerning these graph polynomials.

Some of these results are already published or submitted:

- “The covered components polynomial: A new representation of the edge elimination polynomial”, introducing the covered components polynomial, published as [139],

- “From spanning forests to edge subsets”, relating spanning forest representation and edge subset representation, a preprint is published as [137], submitted to *Ars Mathematica Contemporanea*,
- “Proving properties of the edge elimination polynomial using equivalent graph polynomials”, introducing the subgraph counting polynomial and the trivariate chromatic polynomial, a preprint is published as [138], submitted to *Congressus Numerantium*.

1.3 Organization of This Thesis

This thesis consists of six chapters and one appendix, where Chapter 2 and Chapter 6 provide the necessary terms and a conclusion, respectively. The chapters between are mostly self-contained and can be read in arbitrary order, while the given order is the one suggested.

In Chapter 2, all definitions and notations used throughout the work, with exception of the definition of the graph polynomials, are given.

Then we investigate possible ways to define graph polynomials, *the representations for graph polynomials*, in Chapter 3. The first section is in an extensive but not exhaustive overview about the frameworks used in the literature, expanded by some first classification of them. Relations between representations are given in the second section. The third section is especially devoted to recurrence relations.

In Chapter 4, the edge elimination polynomials are considered, which include the edge elimination polynomial and the graph polynomials equivalent to it. We start with a short introduction of the edge elimination polynomial and then present several equivalent graph polynomials. Their combinatorial interpretations are used to prove some properties valid for all edge elimination polynomials and some relations to other graph polynomials.

A new graph polynomial, the generalized subgraph counting polynomial, is defined in Chapter 5. This proceeds the results of the previous chapter as it generalizes some of them. The main theorem there is the recurrence relation applicable also for hypergraphs, which easily enables to derive many well-known graph polynomials and their recurrence relations from it.

For the sake of convenience, in Appendix A we itemize references and definitions for all mentioned graph polynomials, and in the Glossary the occurrences of the significant terms, including the representations, expansions and graph polynomials, are given.

Chapter 2

Basics in Graph Theory

In this chapter we introduce some graph theory we make use of. Because we discuss a multitude of different graphs polynomials, we touch a lot of miscellaneous areas of graph theory, and, consequently, a long list of definitions and notations is necessary.

While we try to define every term applied, previous knowledge of graph theory as presented in standard textbooks [10; 15; 25; 28; 47; 65; 144] may be advantageous. Readers familiar with this topic may skip to the next chapter.

For the sake of convenience, we use the terms used for graphs also for hypergraphs, for example we speak about “graph polynomial” and “subgraph” instead “hypergraph polynomials” and “subhypergraphs”. Consequently, corresponding theorems will only differ on the assumption of a graph or a hypergraph.

Preliminary, we present the following (non-graph-theoretic) notations: For elements s_1, \dots, s_k , by $\{s_1, \dots, s_k\}$ and $\{s_1, \dots, s_k\}^*$ we denote the *set* and the *multiset* of these elements, and by $|\{s_1, \dots, s_k\}|$ and $|\{s_1, \dots, s_k\}^*|$ we denote their cardinality, respectively. For sets A, B (with $A \subseteq B$), the *interval* $[A, B]$ is the set of subsets of B , which are supersets of A . For a set S , $\binom{S}{k}$ denotes the set of k -element subsets of S . For a statement S , let $[S]$ be equal to 1, if S is true, and 0 otherwise [86].

2.1 Graphs and Hypergraphs

Definition 2.1. A *graph* $G = (V, E)$ is an ordered pair of a set of vertices, the vertex set V , and a multiset of edges, the edge set E , such that each edge is a one- or two-element subset of the vertex set, i.e. $e \in \binom{V}{1} \cup \binom{V}{2}$ for all $e \in E$. An edge $e \in E$ is a *link*, if it is a two-element subsets of V , i.e. $e \in \binom{V}{2}$, and a *loop*, if it is an one-element subset of V , i.e. $e \in \binom{V}{1}$.

Definition 2.2. A *simple graph* is a graph $G = (V, E)$, where each edge is a link and the edge set is a set, i.e. $E \subseteq \binom{V}{2}$.

Definition 2.3. The *edgeless graph* on n vertices, denoted by E_n , is a (simple) graph with n vertices and no edge. The *complete graph* on n vertices, denoted by K_n , is a simple graph with n vertices and the edge set equals the set of two-element subsets of the vertex set.

Definition 2.4. A *hypergraph* $G = (V, E)$ is an ordered pair of a set of vertices, the vertex set V , and a multiset of (hyper)edges, the edge set E , such that each edge is a non-empty subset of the vertex set, i.e. $e \subseteq V$ for all $e \in E$.

Consequently, a graph is a hypergraph $G = (V, E)$, where each edge is a set of at most two vertices: $|e| \leq 2$ for all $e \in E$.

For the sake of convenience, we assume that the vertices are not sets itself, to avoid confusion between a vertex and an edge.

Definition 2.5. Let $G = (V, E)$ be a hypergraph. We refer to the vertex set and to the edge set of G by $V(G)$ and $E(G)$, respectively. A vertex $v \in V$ and an edge $e \in E$ are *incident* (to each other), if $v \in e$. Two edges $e, f \in E$ of G are *adjacent* (to each other), if $e \cap f \neq \emptyset$.

Definition 2.6. Let $G = (V, E)$ be a hypergraph and $v \in V$ a vertex of G . The *degree* of v in G , $\deg(G, v)$, is the number of edges incident to v :

$$\deg(G, v) = |\{e \in E \mid v \in e\}^*|. \quad (2.1)$$

By $\deg^{-1}(G, i)$ we denote the *number of vertices with degree i* in G :

$$\deg^{-1}(G, i) = |\{v \in V \mid \deg(G, v) = i\}|. \quad (2.2)$$

$I(G)$ and $i(G)$ denote the *set of isolated vertices* in G , i.e. the set of vertices with degree 0, and the *number of isolated vertices* in G , respectively:

$$I(G) = \{v \in V \mid \deg(G, v) = 0\}, \quad (2.3)$$

$$i(G) = |I(G)| = \deg^{-1}(G, 0). \quad (2.4)$$

2.2 Homomorphisms and Isomorphisms

Definition 2.7. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. A *homomorphism* from G to G' is a function $f: V \rightarrow V'$, such that for each edge $e \in E$ it holds

$$\bigcup_{v \in e} \{f(v)\} \in E'. \quad (2.5)$$

In other words, a homomorphism maps (the incident vertices of) an edge of G to (the incident vertices of) an edge of G' . Thereby it is possible that no, one or several edges of G are mapped to the same edge of G' :

$$\left\{ \bigcup_{v \in e} \{f(v)\} \mid e \in E \right\} \subseteq E'. \quad (2.6)$$

For the counting of homomorphisms it is usual to also consider which edge of G is mapped to which edge of G' , that means to count functions mapping vertices to vertices and edges to edges.

Definition 2.8. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. The *number of homomorphisms* from G to G' , denoted by $\text{hom}(G, G')$, is defined as

$$\text{hom}(G, G') = \sum_{f: \begin{smallmatrix} V \rightarrow V' \\ E \rightarrow E' \end{smallmatrix}} [\forall v \in V \forall e \in E: v \in e \Rightarrow f(v) \in f(e)]. \quad (2.7)$$

For simple graphs G and G' (in fact if G' has no parallel edges), a homomorphism is given by the function mapping the vertex sets. Hence,

$$\text{hom}(G, G') = \sum_{f: V \rightarrow V'} \left[\forall e \in E: \bigcup_{v \in e} \{f(v)\} \in E' \right]. \quad (2.8)$$

This can be extended to the general case of hypergraphs by considering the number of edges of E' , to which each edge of E can be mapped:

$$\text{hom}(G, G') = \sum_{f: V \rightarrow V'} \prod_{e \in E} \left| \left\{ e' \in E': \bigcup_{v \in e} \{f(v)\} = e' \right\}^* \right|. \quad (2.9)$$

This definition is similar to those used by Garijo, Goodall and Nešetřil [59, Subsection 2.1].

Definition 2.9. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. An *isomorphism* from G into G' is a bijective homomorphism, that is a bijective function $f: V \rightarrow V'$, such that

$$\left\{ \bigcup_{v \in e} \{f(v)\} \mid e \in E \right\}^* = E'. \quad (2.10)$$

The hypergraphs G and G' are *isomorphic*, if there is an isomorphism from G into G' .

In other words, G and G' are isomorphic, if G' can be obtained by a relabeling of the vertices of G .

2.3 Graph Invariants and Graph Polynomials

Definition 2.10. Let \mathcal{G} be the set of hypergraphs and \mathcal{S} some set. A *graph invariant* is a function $f: \mathcal{G} \rightarrow \mathcal{S}$, such that for isomorphic graphs $G, G' \in \mathcal{G}$ it holds

$$f(G) = f(G'). \quad (2.11)$$

Definition 2.11. Let \mathcal{G} be the set of hypergraphs and $\mathbb{R}[x_1, \dots, x_k]$ the ring of polynomials in the commuting variables x_1, \dots, x_k over the real numbers. A *graph polynomial* $P(G, x_1, \dots, x_k)$ is a function $P : \mathcal{G} \rightarrow \mathbb{R}[x_1, \dots, x_k]$.

Definition 2.12. Let \mathcal{G} be the set of hypergraphs. An *invariant graph polynomial* is a graph polynomial, which is a graph invariant, that is a function $P : \mathcal{G} \rightarrow \mathbb{R}[x_1, \dots, x_k]$, such that for isomorphic hypergraphs $G, G' \in \mathcal{G}$ and commuting variables x_1, \dots, x_k it holds

$$P(G, x_1, \dots, x_k) = P(G', x_1, \dots, x_k). \quad (2.12)$$

Until otherwise stated, we consider only graph polynomials, which are also graph invariants, and therefore use “graph polynomial” as abbreviation for “invariant graph polynomial”. In case of $\mathcal{S} = \{0, 1\}$ and $\mathcal{S} = \mathbb{N}$ one usually speaks about *(invariant) graph properties* and *(invariant) graph parameters*, respectively. While we consider polynomial rings over the real numbers for the definition, the coefficients of the graph polynomials investigated in the following are integers. Furthermore, all variables are commuting, and in case of multivariate polynomials we define $X = (x_1, \dots, x_k)$ and write $\mathbb{R}[X]$ and $P(G, X)$ instead of $\mathbb{R}[x_1, \dots, x_k]$ and $P(G, x_1, \dots, x_k)$, respectively. In particular, by $P(G, X)$ we denote an arbitrary graph polynomial. Additionally, we use $P(G, X, y)$ for a graph polynomial in the variables x_1, \dots, x_k, y .

Definition 2.13. Let \mathcal{G} be the set of hypergraphs and $P(G, X), P'(G, X)$ two graph polynomials. $P(G, X)$ and $P'(G, X)$ are *equivalent* (to each other), if there is a bijection $f : \mathbb{R}[X] \rightarrow \mathbb{R}[X]$, such that for all graphs $G \in \mathcal{G}$ it holds

$$P(G, X) = f(P'(G, X)). \quad (2.13)$$

Definition 2.14. Let $P = P(G, X)$ be a (graph) polynomial with

$$P = \sum_{i_1, \dots, i_k} a_{i_1, \dots, i_k} x_1^{i_1} \cdots x_k^{i_k}, \quad (2.14)$$

where $i_1, \dots, i_k \in \mathbb{N}$ and $a_{i_1, \dots, i_k} \in \mathbb{R}$. We denote by $\deg_x(P)$ the degree of x in P and by $[x_j^l](P)$ the sum of all monomials including the variable x_j in the power l , i.e.

$$[x_j^l](P) = \sum_{\substack{i_1, \dots, i_k \\ i_j = l}} a_{i_1, \dots, i_k} x_1^{i_1} \cdots x_{j-1}^{i_{j-1}} x_{j+1}^{i_{j+1}} \cdots x_k^{i_k}. \quad (2.15)$$

Furthermore, we expand this to several variables and write $[x_1^{i_1} \cdots x_j^{i_j}](P)$ instead of $[x_1^{i_1}](\cdots [x_j^{i_j}](P) \cdots)$.

For a graph polynomial $P(G, x)$ in a single variable x , $[x^i](P(G, x))$ is the coefficient of x^i in $P(G, x)$.

2.4 Subgraphs and Components

Definition 2.15. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. G' is a *subgraph* of G , denoted by $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. G' is a *proper subgraph* of G , denoted by $G' \subset G$, if additionally $V' \subset V$ or $E' \subset E$.

We say G (properly) contains G' and G' is (properly) contained in G .

Definition 2.16. Let $G = (V, E)$ be a hypergraph and $A \subseteq E$ an edge subset of G . The *spanning subgraph* $G\langle A \rangle$ is the graph

$$G\langle A \rangle = (V, A). \quad (2.16)$$

We say $G\langle A \rangle$ is the subgraph spanned by A . The spanning subgraph $G\langle A \rangle$ is the subgraph obtained from G by deleting the edges of $E \setminus A$.

Definition 2.17. Let $G = (V, E)$ be a hypergraph and $W \subseteq V$ a vertex subset of G . The *induced subgraph* $G[W]$ is the graph

$$G[W] = (W, \{e \in E \mid e \subseteq W\}^*). \quad (2.17)$$

We say $G[W]$ is the subgraph of G induced by W and W induces $G[W]$ in G . The induced subgraph $G[W]$ is the subgraph obtained from G by deleting the vertices of $V \setminus W$.

Definition 2.18. Let $G = (V, E)$ be a hypergraph and $A \subseteq E$ an edge subset of G . The *edge-induced subgraph* $G[A]$ is the graph

$$G[A] = \left(\bigcup_{e \in A} e, A \right). \quad (2.18)$$

We say $G[A]$ is the subgraph of G edge-induced by A , A edge-induces $G[A]$ in G . The edge-induced subgraph $G[A]$ is the subgraph obtained from G by first deleting the edges of $E \setminus A$ and then deleting all isolated vertices. In particular, $G[E]$ is the graph G with all isolated vertices removed.

Definition 2.19. Let $G = (V, E)$ be a hypergraph. A *component* of G is a subgraph $G' = (V', E')$, such that for each edge $e \in E$ either $e \in E'$ or $e \cap V' = \emptyset$. A *connected component* of G is a non-empty component of G minimal with respect to inclusion. The *number of connected components* of G is denoted by $k(G)$. If $k(G) = 1$, then G is *connected*.

Definition 2.20. Let $G = (V, E)$ be a hypergraph. A *covered component* of G is a component of G including at least one edge. A *covered connected component* of G is a connected components of G including at least one edge. The *number of covered connected components* of G is denoted by $c(G)$.

Remarks

Please note that especially for the different kind of subgraphs the notation is not uniform in different textbooks, see [11, Section 1.1; 25, Section I.1; 28, Section 2.1, 2.2; 47, Section 1.1; 65, Section 1.2; 144, Section I.3].

2.5 Cycles, Forests and Spanning Forests

Definition 2.21. Let $G = (V, E)$ be a graph. G is *cyclic*, if G has a subgraph $G' = (V', E')$ including at least one edge, such that for each edge $e \in E'$ of G' it holds

$$k(G') = k(G'_{-e}). \quad (2.19)$$

Otherwise, G is *acyclic*.

Definition 2.22. Let $G = (V, E)$ be a graph. G is a *cycle*, if G is cyclic and has no proper cyclic subgraph. G is a *forest*, if G is acyclic. G is *tree*, if G is acyclic and connected.

Definition 2.23. Let $G = (V, E)$ be a graph and $A \subseteq E$ an edge subset of G . A tree $T = G\langle A \rangle = (V, A)$ is a *spanning tree* of G . The *set of all spanning trees* of G is denoted by $\mathcal{T}(G)$.

Definition 2.24. Let $G = (V, E)$ be a graph and $A \subseteq E$ an edge subset of G . A forest $F = G\langle A \rangle = (V, A)$ is a *spanning forest* of G , if $k(G) = k(F)$. The *set of all spanning forests* of G is denoted by $\mathcal{F}(G)$.

Remarks

While the term “spanning tree” is unambiguous, the term “spanning forest” is not, because not every spanning subgraph which is a forest is a “spanning forest” [25, Section X.5]. A spanning forest is the union of spanning trees for each connected component.

2.6 Graph Operations

Definition 2.25. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. The *union* $G \cup G'$ is the graph arising from the union of the vertex sets and edge sets:

$$G \cup G' = (V \cup V', E \cup E'). \quad (2.20)$$

Definition 2.26. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. The *intersection* $G \cap G'$ is the graph arising from the intersection of the vertex sets and edge sets:

$$G \cap G' = (V \cap V', E \cap E'). \quad (2.21)$$

Definition 2.27. Let $G = (V, E)$ and $G' = (V', E')$ be hypergraphs. The *disjoint union* $G \cup G'$ is the graph arising from the union of disjoint copies of both graphs.

In other words, first (the vertices of) the graphs are relabeled, such that the intersection of both graphs is empty, and then the union is formed.

Definition 2.28. Let $G = (V, E)$ be a hypergraph and $e \in E$ an edge of G . We define the following *edge operations*:

- $-e$: *deletion* of the edge e , i.e. e is removed,
- $/e$: *contraction* of the edge e , i.e. e is removed and its incident vertices are merged (parallel edges and loops may occur),
- $\dagger e$: *extraction* of the edge e , i.e. the vertices incident to e and their incident edges (including e itself) are removed.

The arising hypergraphs are denoted by G_{-e} , $G_{/e}$ and $G_{\dagger e}$, respectively.

Definition 2.29. Let $G = (V, E)$ be a hypergraph and $e \subseteq V$ a possible edge of G (with respect to the type of graph). We define the following *non-edge operations*:

- $+e$: *insertion* of the edge e , i.e. e is added,
- $/e$: *contraction* of the vertices in e , i.e. the vertices in e are merged (parallel edges and loops may occur),

The arising hypergraphs are denoted by G_{+e} and $G_{/e}$, respectively.

Definition 2.30. Let $G = (V, E)$ be a hypergraph, $v \in V$ a vertex and $W \subseteq V$ a vertex subset of G . We define the following *vertex operations*:

- $\ominus v$: *deletion* of the vertex v , i.e. v and its incident edges are removed,
- $\ominus W$: *deletion* of all vertices in the vertex subset W , i.e. all vertices $v \in W$ and their incident edges are removed.

The arising hypergraphs are denoted by $G_{\ominus v}$ and $G_{\ominus W}$, respectively.

Remarks

We use $\ominus W$ instead of the usual $-W$ for the deletion of the vertex set W from a graph $G = (V, E)$, because each edge is also a vertex subset, and therefore with $W = e \in E$ we have to distinguish between the deletion of the edge e , $-e$, and the deletion of the vertices incident to the edge e , $\ominus e$. For probably similar reasons this notation is already used in the literature [57; 80]. It holds $G_{\dagger e} = G_{\ominus e}$, but the first term is defined only for $e \in E$, whereas the second one is defined for any $e \subseteq V$.

The defined edge and vertex operations are known from the recurrence relations for the chromatic polynomial, see Section 3.3, the matching polynomial [56, Theorem 1] and the independence polynomial [68, Proposition 4].

There are several more graph operations that can be found in the literature, for example:

- *vertex contraction* [134, Section 5.1],
- *insertion of “pseudo-edges”* [9, Section 1],
- *Kellman’s operation* (for adjacent vertices) [43, Definition 2.0.1]
- *NA-Kellman’s operation* (for non-adjacent vertices) [43, Definition 2.8.1],
- *adaption of two vertices* [34, Section 3],
- *cloning of edges and vertices* [78, Section 3].

2.7 Graphs with a Linear Order on the Edge Set

In the following we consider graphs $G = (V, E)$ with a linear order $<$ on the edge set E . This linear order can be represented by a bijection $\beta: E \rightarrow \{1, \dots, |E|\}$ for all $e, f \in E$ with

$$e < f \Leftrightarrow \beta(e) < \beta(f). \quad (2.22)$$

Definition 2.31 (Section 7 in [152]). Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . Let $C = (V_C, E_C) \subseteq G$ be a cycle and $e \in E_C$ the maximal edge of C with respect to $<$. Then $E_C \setminus \{e\}$ is a *broken cycle* in G with respect to $<$. The set of all broken cycles of G with respect to $<$ is denoted by $\mathcal{B}(G, <)$.

Definition 2.32 (Section 3 in [141]). Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F = (V, A) \in \mathcal{F}(G)$ a spanning forest of G . An edge $e \in A$ is *internally active* in F with respect to G and $<$, if there exists no edge $f \in E \setminus A$, such that $e < f$ and $F_{-e+f} \in \mathcal{F}(G)$. We denote the set of internally active edges and the number of internally active edges of F with respect to G and $<$ by $E_i(F, G, <)$ and $i(F, G, <)$, respectively.

An edge e in the spanning forest F is internally active, if it is the maximal edge of all edges in the cut crossed by e itself (connecting the vertices in the connected components arising by deleting e from F). In other words, the edge e can not be replaced by a greater edge (not in the spanning forest), such that F remains a spanning forest. Hence, formally we have

$$E_i(F, G, <) = \{e \in E(F) \mid \nexists f \in E(G) \setminus E(F): e < f \wedge F_{-e+f} \in \mathcal{F}(G)\}. \quad (2.23)$$

Definition 2.33 (Section 3 in [141]). Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F = (V, A) \in \mathcal{F}(G)$ a spanning forest of G . An edge $f \in E \setminus A$ is *externally active* in F with respect to G and $<$, if there exists no edge $e \in A$, such that $f < e$ and $F_{-e+f} \in \mathcal{F}(G)$. We denote the *set of externally active edges* and the *number of externally active edges* of F with respect to G and $<$ by $E_e(F, G, <)$ and $e(F, G, <)$, respectively.

An edge f not in the spanning forest F is externally active, if it is the maximal edge of all edges in the cycle closed by f itself (in the cycle arising by inserting f into F). In other words, there is no greater edge (in the spanning forest), which can be replaced by f , such that F remains a spanning forest. Hence, formally we have

$$E_e(F, G, <) = \{f \in E(G) \setminus E(F) \mid \nexists e \in E(F) : f < e \wedge F_{-e+f} \in \mathcal{F}(G)\}. \quad (2.24)$$

Remarks

In the literature, instead of “broken cycle” often the term “broken circuit” is used, also in Whitney’s original definition [151, Section 2; 152, Section 7]. Whitney uses the term “circuit” for what we call “cycle”, and from this by deleting the maximal edge (in his words “dropping out the last arc” with respect to a “definite order” [151, Section 2]) he came to a “broken circuit”. By the same analogy we get a “broken cycle”.

Furthermore, in the literature a broken cycle is often defined as a subgraph, here it is given as an edge subset. Whitney’s own definitions are not explicit in this way and allow both.

Broken-cycle-free edge subsets, same as broken cycles, were first considered by Whitney and result in the well-known Broken-cycle Theorem [151, Theorem 1], which states a combinatorial interpretation of the coefficients of the chromatic polynomial. We present and extend this result in Subsection 3.1.4.

Internally and externally active edges were first used by Tutte [141] to state the dichromate, nowadays called Tutte polynomial. For some background to the definition of internally and externally active edges and the Tutte polynomial, see [8; 92; 145].

Broken cycles and externally active edges are related as follows: A spanning forest of G is externally active, if and only if it includes a broken cycle [8, Section 4].

2.8 Partitions of Graphs

Definition 2.34. Let S be a set. A *partition* π of S is a family of non-empty disjoint subsets of S , such that their union is S . The elements of π are called *blocks* and the *number of blocks* of π is denoted by $|\pi|$. The *set of partitions* of S is denoted by $\Pi(S)$.

Definition 2.35. Let $G = (V, E)$ be a graph. A *(vertex) partition* of G is a partition $\pi \in \Pi(V)$ of the vertex set V . The *set of (vertex) partitions* of G is denoted by $\Pi(G)$.

Definition 2.36. Let $G = (V, E)$ be a graph. A *connected partition* of G is a partition $\pi \in \Pi(G)$ of the vertex set V , such that $G[W]$ is connected for all $W \in \pi$, that is the subgraph induced by the vertices of each block is connected. The *set of connected partitions* of G is denoted by $\Pi_c(G)$.

Definition 2.37. Let $G = (V, E)$ be a graph. An *independent partition* of G is a partition $\pi \in \Pi(G)$ of the vertex set V , such that $G[W]$ is edgeless for all $W \in \pi$, that is the subgraph induced by the vertices of each block is edgeless. The *set of independent partitions* of G is denoted by $\Pi_i(G)$.

Let \leq be the usual refinement relation for partitions. Then $(\Pi_c(G), \leq)$ is a poset and even more it is a lattice, known as *bond lattice* [118].

Definition 2.38. Let $G = (V, E)$ be a graph and $\Pi_c(G)$ the set of all connected partitions of G . We denote the *Möbius function* and the *minimal element* of the lattice $(\Pi_c(G), \leq)$ as $\mu_{\Pi_c(G)} = \mu$ and $\hat{0}_{\Pi_c(G)} = \hat{0}$, respectively.

2.9 Colorings, Independent Sets and Matchings

Definition 2.39. Let $G = (V, E)$ be a hypergraph. A *(vertex) coloring* of G is a function from the vertex set V in some set C , whose elements are referred to as *colors*. A *k-coloring* of G is a function $\phi: V \rightarrow \{1, \dots, k\}$. A *monochromatic edge* of G with respect to some *k-coloring* ϕ is an edge $e \in E$, such that all vertices incident to e are mapped to the same color. A *proper k-coloring* of G is a *k-coloring* without any monochromatic edges.

Definition 2.40. Let $G = (V, E)$ be a hypergraph. An *independent (vertex) set* of G is a vertex subset $W \subseteq V$ of G , such that $e \not\subseteq W$ for all edges $e \in E$.

Definition 2.41. Let $G = (V, E)$ be a hypergraph. A *matching* of G is an edge subset $A \subseteq E$, such that for all different edges $e, f \in A$ it holds $e \cap f = \emptyset$.

Remarks

It would be also possible to generalize proper colorings to hypergraphs by requiring that any two vertices incident to the same edge should be colored differently. With respect to proper colorings, this would be equivalent to substitute each hyperedge for a set of edges connecting any two vertices incident to the hyperedge [131, Footnote 2 on page 281].

2.10 Reconstructability

The *reconstruction conjecture* of Kelly [85] and Ulam [146] states that every graph $G = (V, E)$ with at least three vertices can be reconstructed from (the isomorphism classes) of its *deck* $\mathcal{D}(G)$, which is the multiset of (isomorphism classes) of vertex-deleted subgraphs, i.e. $\mathcal{D}(G) = \{G_{\ominus v} \mid v \in V\}^*$.

This question can be “restricted” to a graph polynomial $P(G, X)$ as follows: Can the graph polynomial of a given graph be reconstructed from the graph polynomials of its deck?

Definition 2.42. Let $G = (V, E)$ be a graph and $P(G, X)$ a graph polynomial. The *polynomial deck* $\mathcal{D}_P(G)$ is the multiset

$$\mathcal{D}_P(G) = \{P(G_{\ominus v}, X) \mid v \in V\}^*. \quad (2.25)$$

A graph polynomial is *reconstructable from the polynomial deck*, if $P(G, X)$ can be determined from $\mathcal{D}_P(G)$.

2.11 Reliability Domination

Definition 2.43 (Equation (2.3) in [81]). Let $G = (V, E)$ be a graph, $A \subseteq E$ an edge subset of G and $k \in \mathbb{N}$. For all edge subsets $B \subseteq A$, the *signed domination* $d(G, B, k)$ is recursively defined by

$$[k(G\langle A \rangle) \leq k] = \sum_{B \subseteq A} d(G, B, k). \quad (2.26)$$

Theorem 2.44 (Theorem 4.2 in [81]). Let $G = (V, E)$ be a graph, $A \subseteq E$ an edge subset of G and $k \in \mathbb{N}$. The signed domination $d(G, A, k)$ satisfies

$$d(G, A, k) = \sum_{B \subseteq A} (-1)^{|A| - |B|} [k(G\langle B \rangle) \leq k]. \quad (2.27)$$

Proof. The statement follows directly by Möbius inversion. \square

Definition 2.45 (Proposition 2.8 in [82]). Let $G = (V, E)$ be a graph, $A \subseteq E$ an edge subset of G and $k \in \mathbb{N}$. A k -forest of $G\langle A \rangle$ is a spanning subgraph $G\langle B \rangle$, such that $G\langle B \rangle$ is a forest, $B \subseteq A$ and $k(G\langle B \rangle) \leq k$. We denote the set of k -forests of $G\langle A \rangle$ by $\mathcal{F}(G, A, k)$. A k -formation D of $G\langle A \rangle$ is a non-empty set of k -forests of $G\langle A \rangle$, such that their union is $G\langle A \rangle$, i.e. a non-empty subset $F \subseteq \mathcal{F}(G, A, k)$ is a k -formation of $G\langle A \rangle$, if $\bigcup_{H \in F} H = G\langle A \rangle$. The set of k -formations of $G\langle A \rangle$ is denoted by $\mathcal{D}(G, A, k)$. The *signed domination* $d'(G, A, k)$ is defined as the number of k -formations of $G\langle A \rangle$ of odd cardinality minus the number of k -formations of $G\langle A \rangle$ of even cardinality, i.e.

$$d'(G, A, k) = \sum_{D \in \mathcal{D}(G, A, k)} (-1)^{|D| - 1}. \quad (2.28)$$

Remarks

Reliability domination has been defined in order to find a combinatorial interpretation of the coefficients of the reliability polynomial $R(G, p)$ [124, Equation (7)].

We give two different definitions for signed domination and show their equivalence in Subsection 3.2.3. Due to Satyanarayana and Tindell [125, Section 1], the original definition of signed domination is given by Satyanarayana [123] in terms of “formations” of a graph. It is often defined with respect to a vertex subset, this is considered in many publications by Satyanarayana and his coauthors, see [24; 116] and the references therein. Another definition was given by Huseby [81; 82] for “clutters”. Both definitions above orient more on the last one in the case of graphs with respect to the number of connected components.

Signed domination is also related to spanning trees: The number of spanning trees having no externally active edge equals the absolute value of the signed domination $d(G, E, 1)$ [23, Corollary 4.2].

Chapter 3

Representations for Graph Polynomials

This chapter is devoted to the multitude of different ways and frameworks applied to define a graph polynomial — to the *representations for graph polynomials*.

In Section 3.1 we give a survey on different representations used in the literature. The list does not claim to be exhaustive. However, we hope to mention the main representatives available in the literature. For each we give an informal definition, an exemplary graph polynomial defined in this framework, and a corresponding formulation of the chromatic polynomial. Thereby we introduce several well-known graph polynomials and expansions of the chromatic polynomial.

Some results that serve as a link between the edge subset representation and some other representations are presented in Section 3.2. With exception of the generalization of the Broken-cycle Theorem, the statements are, in principle, already known. However, these statements provide good examples for non-obvious relations between different representations of graph polynomials and either the proofs, for instance for the relation to reliability domination representation, or the applications, for instance for the relation to spanning forest representation, seem to be new.

In Section 3.3 we discuss recurrence relations, used to define graph polynomials or satisfied by them, in more detail. We show some examples and prove two general results.

3.1 Overview and Definitions

There are various ways to define graph polynomials and in this section we introduce the main patterns we found (with names wherever possible also from the literature). The given itemization is neither complete nor are the given representations formally defined or disjoint. We also mix between how the graph

polynomials are written down and in which graph theoretic terms these are defined.

For “how the graph polynomials are written down” there are in fact two possibilities: value representation, where the number of the counted objects equals the polynomial at a given value, and generating function representation, where the number of counted objects equals a coefficient of a monomial in the polynomial. There are also graph polynomials which combine both, for example the bad coloring polynomial which counts for a given number of colors (the variable x) the number of some edges as a generating function (in the variable z).

For the graph theoretic terms used, there are some more possibilities, which can be grouped as counting subgraphs, counting mappings and others.

For counting subgraphs, most often spanning subgraphs, given by edge subsets, or induced subgraphs, given by a vertex subsets, are considered. Broken-cycles, reliability domination and spanning forests are in fact defined in terms of edge subsets, but we list them as single items because of their relevance.

When mappings are counted, there are again three possibilities: mappings of the vertex set, mappings of the edge set, and homomorphisms to some graph, which are in fact mappings of the vertex and edge set. Spin models (mostly used for graph polynomials defined in physics) and colorings are in fact the same kind of vertex mappings, in the first case the vertices are mapped to a set of “spins”, in the second to a set of “colors”. Consequently, both differ only in their “language”, not in the mathematics behind. We add a superior category denoted “edge mapping representations” to make clear what is meant by a “vertex model”.

The three “other” representations are using matrices and matroids associated to the graph or recurrence relations.

All together, we classify the representations for graph polynomials as follows:

- value representation,
- generating function representation,
- subgraph representation:
 - edge subset representation,
 - broken-cycle representation,
 - reliability domination representation,
 - spanning forest representation,
 - vertex subset representation,
- vertex mapping representation:
 - spin model,
 - coloring representation,
 - partition representation,

- edge mapping representation:
 - vertex model,
- homomorphism representation.
- matrix representation,
- matroid representation,
- recurrence relation representation,

We continue by introducing the representations, a characteristic graph polynomial using this representation, and a corresponding expansion of the chromatic polynomial one by one. The thereby given list of such expansions is not complete, missing are, amongst others, some subgraph expansions concerning only special subgraphs [14; 103].

In the following, we assume $G = (V, E)$ to be a graph with some linear order $<$ on the edge set E . (While most of the definitions of graph polynomials also make sense in the case of hypergraphs, some expansion would not be valid.)

3.1.1 Value Representation

A *value representation* states a graph polynomial by a combinatorial interpretation for given values (mostly integers) of the variables.

The chromatic polynomial $\chi(G, x)$ (for $x \in \mathbb{N}$) is defined [18; 52] as the number of proper (vertex) colorings of G with (at most) x colors,

$$\chi(G, x) = |\{\text{proper colorings of } G \text{ with } x \text{ colors}\}|. \quad (3.1)$$

In fact, from this definition it is not obvious that $\chi(G, x)$ is a polynomial in x .

3.1.2 Generating Function Representation

A *generating function representation* states a graph polynomial as the generating function for a number sequence.

The matching polynomial $M(G, x, y)$ is defined [56; 64] as the generating function of the number of matchings with respect to their cardinality,

$$M(G, x, y) = \sum_i a_i x^{|V| - |\bigcup_{e \in A} e|} y^i, \quad (3.2)$$

where a_i is the number of matchings of G with cardinality i .

The chromatic polynomial $\chi(G, x)$ can be defined [151, Theorem 1; 52, Theorem 2.3.1] as the generating function

$$\chi(G, x) = \sum_i m_i x^{|V| - i}, \quad (3.3)$$

where $(-1)^i m_i$ is the number of spanning subgraphs of G with i edges not containing any broken cycle (with respect to $<$).

3.1.3 Edge Subset Representation

An *edge subset representation* states a graph polynomial as a sum over edge subsets.

The Potts model $Z(G, x, y)$ is defined [129] as

$$Z(G, x, y) = \sum_{A \subseteq E} x^{k(G \langle A \rangle)} y^{|A|}. \quad (3.4)$$

The chromatic polynomial $\chi(G, x)$ has the edge subset expansion [151, Section 2; 52, Theorem 2.2.1]

$$\chi(G, x) = \sum_{A \subseteq E} (-1)^{|A|} x^{k(G \langle A \rangle)}. \quad (3.5)$$

3.1.4 Broken-cycle Representation

A *broken-cycle representation* states a graph polynomial as a sum over broken-cycle-free edge subsets.

The chromatic polynomial $\chi(G, x)$ has the broken-cycle expansion [152, Section 7; 52, Theorem 2.3.1]

$$\chi(G, x) = \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}(G, <): B \not\subseteq A}} (-1)^{|A|} x^{|V| - |A|}. \quad (3.6)$$

3.1.5 Reliability Domination Representation

A *reliability domination representation* states a graph polynomial in terms of reliability domination.

The reliability polynomial $R(G, p)$ can be defined [124, Equation (7)] as

$$R(G, p) = \sum_{A \subseteq E} d(G, A, 1) p^{|A|}. \quad (3.7)$$

The chromatic polynomial $\chi(G, x)$ has the reliability domination expansion [125, Proposition 3.3, due to Rodriguez]

$$\chi(G, x) = (-1)^{|E|} (1 - x) \sum_{k=1}^{|V|-1} d(G, E, k) x^k. \quad (3.8)$$

3.1.6 Spanning Forest Representation

A *spanning forest representation* states a graph polynomial as a sum over spanning forests. In case of connected graphs we have spanning trees and therefore speak about *spanning tree representation*.

The Tutte polynomial $T(G, x, y)$ is defined [141, Section 3; 15, Definition 13.6] as

$$T(G, x, y) = \sum_{F \in \mathcal{F}(G)} x^{i(F, G, <)} y^{e(F, G, <)}. \quad (3.9)$$

The chromatic polynomial $\chi(G, x)$ has the spanning forest expansion [141, Equation (4) and (21); 15, Theorem 14.1]

$$\chi(G, x) = (-1)^{|V|} (-x)^{k(G)} \sum_{\substack{F \in \mathcal{F}(G) \\ e(F, G, <) = 0}} (1 - x)^{i(F, G, <)}. \quad (3.10)$$

3.1.7 Vertex Subset Representation

A *vertex subset representation* states a graph polynomial as a sum over vertex subsets.

The independence polynomial $I(G, x)$ is defined [68; 94] as

$$I(G, x) = \sum_{W \subseteq V} [W \text{ is independent set in } G] x^{|W|}. \quad (3.11)$$

The chromatic polynomial $\chi(G, x)$ (for $x \in \mathbb{N}$) has the “recursive vertex subset expansion” [54, Theorem 9.7.17]

$$\chi(G, x) = \sum_{W \subseteq V} [W \text{ is independent set in } G] \chi(G_{\ominus W}, x - 1). \quad (3.12)$$

3.1.8 Spin Model

A *spin model* states a graph polynomial as a sum over mappings of the vertex set in a set, whose elements are called “spins” or “states”, therefore also the name *state model* is common. The representation has its origin in mathematical physics, but is in fact equivalent to counting colorings (coloring representation). See also [70; 105; 126].

The extended Negami polynomial $\tilde{f}(G, t, x, y, z)$ (for $t \in \mathbb{N}$) can be defined [105, page 327] as

$$\tilde{f}(G, t, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, t\}} \prod_{e \in E} w(e), \quad (3.13)$$

where

$$w(e) = \begin{cases} x + y & \text{if } \forall v \in e: \phi(v) = 1, \\ z + y & \text{if } \exists c \neq 1 \forall v \in e: \phi(v) = c, \\ y & \text{if } \nexists c \forall v \in e: \phi(v) = c. \end{cases} \quad (3.14)$$

The chromatic polynomial $\chi(G, x)$ (for $x \in \mathbb{N}$) has the spin model expansion [70, page 209]

$$\chi(G, x) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{e \in E} \gamma(e), \quad (3.15)$$

where

$$\gamma(e) = \begin{cases} 0 & \text{if } \exists c: \forall v \in e: \phi(v) = c, \\ 1 & \text{if } \nexists c: \forall v \in e: \phi(v) = c. \end{cases} \quad (3.16)$$

Please observe, that Noble and Welsh use the term “states model representation” for an equation of the weighted graph polynomial given as a sum over edge subsets [107, Theorem 4.3].

Spin models may be used to count isomorphisms and therefore to define complete graph invariants [70, Proposition on page 213].

3.1.9 Coloring Representation

A *coloring representation* (*vertex coloring model*) states a graph polynomial by counting colorings. It has its origin in graph theory, but is in fact equivalent to spin models.

The bivariate chromatic polynomial $P(G, x, y)$ (for $x, y \in \mathbb{N}$) is defined as the number of (vertex) colorings with (at most) x colors, such that the vertices incident to each monochromatic edge are colored by a color $c > y$ [50, Section 1],

$$P(G, x, y) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \leq y \forall v \in e: \phi(v) = c}} 0. \quad (3.17)$$

The “coloring expansion” of the chromatic polynomial is exactly its definition: The chromatic polynomial $\chi(G, x)$ (for $x \in \mathbb{N}$) is defined as

$$\chi(G, x) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \forall v \in e: \phi(v) = c}} 0. \quad (3.18)$$

The connection between graph polynomials “counting generalized colorings” and graph polynomials definable in Second Order Logic (SOL) is examined by Kotek, Makowsky and Zilber [90; 91].

3.1.10 Partition Representation

A *partition representation* states a graph polynomial as a sum over set partitions, usually over (special) partitions of the vertex set.

The partition polynomial $Q(G, x)$ is defined [127, Section 4] as

$$Q(G, x) = \sum_{\pi \in \Pi_c(G)} x^{|\pi|}, \quad (3.19)$$

and the adjoint polynomial $h(G, x)$ (for a simple graph $G = (V, E)$) is defined [52, Section 11.1] as

$$h(G, x) = \sum_{\pi \in \Pi_i(\bar{G})} x^{|\pi|}, \quad (3.20)$$

where $\bar{G} = (V, \binom{V}{2} \setminus E)$.

The chromatic polynomial $\chi(G, x)$ has the connected partition expansion [118, Equation (*) in Section 9] and the independent partition expansion [52, Theorem 1.4.1]

$$\chi(G, x) = \sum_{\pi \in \Pi_c(G)} \mu(\hat{0}, \pi) \cdot x^{|\pi|}, \quad (3.21)$$

$$= \sum_{\pi \in \Pi_i(G)} x^{|\pi|}. \quad (3.22)$$

3.1.11 Vertex Model

A *vertex model* states a graph polynomial as a sum over mappings of the edge set in a set, whose elements are called “states”. The value of each such mapping is determined by the images of the edges incident to each vertex. This representation is in fact the opposite of a spin model, as the roles of vertices and edges are interchanged. Just as there, we can consider the states as colors, therefore also the names *edge model* and *edge coloring model* are used, see [70; 132].

The edge coloring polynomial $\chi'(G, x)$ is defined [70, Section 2] as

$$\chi'(G, x) = \sum_{\phi: E \rightarrow \{1, \dots, x\}} \prod_{v \in V} \gamma(v), \quad (3.23)$$

with

$$\gamma(v) = \begin{cases} 0 & \text{if } \exists e_1, e_2 \in E: v \in e_1 \cap e_2 \vee \phi(e_1) = \phi(e_2), \\ 1 & \text{otherwise.} \end{cases} \quad (3.24)$$

Spin model and vertex model can be related via the line graph [70, Subsection 2.3]. Consequently, the chromatic polynomial of a line graph has the “vertex model expansion”

$$\chi(L(G), x) = \chi'(G, x). \quad (3.25)$$

3.1.12 Homomorphism Representation

A *homomorphism representation* defines a graph polynomial of a graph by counting its homomorphisms to some graphs.

In general, for a class \mathcal{H} of graphs and a function $w: \mathcal{H} \rightarrow \mathbb{R}$, a graph polynomial $H(G, \mathcal{H}, w)$ can be defined as the weighted sum of the number of homomorphisms from G to the graphs in \mathcal{H} :

$$H(G, \mathcal{H}, w) = \sum_{H \in \mathcal{H}} w(H) \cdot \text{hom}(G, H). \quad (3.26)$$

The homomorphism polynomial by Garijo et al. $H(G, k, x, y, z)$ (for $k, x, y, z \in \mathbb{N}$) is defined [59, page 1044] as

$$H(G, k, x, y, z) = \text{hom}(G, H_{k,x,y,z}), \quad (3.27)$$

where the graph K_k^l is a complete graph on k vertices with l loops attached at each vertex and the graph $H_{k,x,y,z}$ arises by the join of a K_k^z with the disjoint union of y copies of K_x^z .

The chromatic polynomial $\chi(G, x)$ (for $x \in \mathbb{N}$) has the homomorphism expansion [72, Proposition 1.7]

$$\chi(G, x) = \text{hom}(G, K_x). \quad (3.28)$$

There are other “homomorphism polynomials”, for example one defined originally by Bari for simple graphs [9; 62] and extended by Gillman to graphs [62], that count homomorphisms of a graph to its subgraphs.

Nowadays, homomorphisms of graphs and polynomials counting them seem to get increasing attention [59; 60; 72].

3.1.13 Matrix Representation

A *matrix representation* states a graph polynomial as a function of a matrix. A good overview of graph polynomials defined as determinants and permanents of matrices (related to the adjacency matrix) is given by Parthasarathy [113, Subsection 2.1 and Section 5].

The characteristic polynomial $\phi(G, x)$ is defined [44] as the characteristic polynomial of the adjacency matrix $A(G)$,

$$\phi(G, x) = \det(xI - A(G)), \quad (3.29)$$

where $A(G) = [a_{u,v}]_{u,v \in V}$ with $a_{u,v} = |\{e \in E \mid e = \{u, v\}\}|$ and I is the identity matrix of format $|V| \times |V|$.

While the chromatic polynomial of an arbitrary graph can be written in a matrix equation, each of these in fact uses another representation (to represent the entries of the matrix). For special (symmetric) graphs there is a “matrix method” to calculate the values of their chromatic polynomials [16; 17].

3.1.14 Matroid Representation

A *matroid representation* states a graph polynomial as a function of a matroid. For the corresponding definitions we refer to [110].

The rank-generating function $S(G, x, y)$ can be defined [33, Section 6.2; 25, Section X.1] as

$$S(G, x, y) = \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-r(A)}, \quad (3.30)$$

where $r(A)$ is the rank of the set A in the cycle matroid of G .

The chromatic polynomial $\chi(G, x)$ has the matroid representation [33, Proposition 6.3.1]

$$\chi(G, x) = x^{k(G)} (-1)^{V-k(G)} \sum_{A \subseteq E} (-x)^{r(E)-r(A)} (-1)^{|A|-r(A)}, \quad (3.31)$$

where $r(A)$ is the rank of the set A in the cycle matroid of G .

3.1.15 Recurrence Relation Representation

A *recurrence relation representation* states a graph polynomial by recurrence relations satisfied.

The edge elimination polynomial $\xi(G) = \xi(G, x, y, z)$ is defined [4, Equation (13)] as

$$\xi(G) = \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{+e}), \quad (3.32)$$

$$\xi(G^1 \cup G^2) = \xi(G^1) \cdot \xi(G^2), \quad (3.33)$$

$$\xi(K_1) = x. \quad (3.34)$$

The chromatic polynomial $\chi(G, x)$ satisfies the recurrence relations [153, due to Foster; 52]

$$\chi(G, x) = \chi(G_{-e}) - \chi(G_{/e}), \quad (3.35)$$

$$\chi(G^1 \cup G^2) = \chi(G^1) \cdot \chi(G^2), \quad (3.36)$$

$$\chi(K_1) = x. \quad (3.37)$$

Recurrence relations are discussed in more detail in Section 3.3.

3.2 Relations to the Edge Subset Representation

We have already seen that some representations are related to each other, for instance the spin model and the coloring representation are equivalent. Some other relations are already given implicitly by the different expansions of the chromatic polynomial. A generalization of this graph polynomial is the Potts model, which is defined in graph theory (as given in Equation (3.4)) by a sum

over edge subsets and in mathematical physics by a spin model [128; 129]. It therefore already gives a more general relation of those two representations than the expansions of the chromatic polynomial.

In this section we look for relations between different representations which do not assume any special graph polynomial, but hold for all graph polynomials satisfying some properties. The required properties are linked to a function ranging over edge subsets, consequently such an expansion is necessary for the application of the statement. In fact, each of these relations is a generalization of the results known for the chromatic polynomial and hence this graph polynomial (and the function in its edge subset expansion) fulfills the requirements.

3.2.1 Broken-cycle Representation

The well-known Broken-cycle Theorem can be extended to link between edge subset representations and broken-cycle representations.

Please remember that broken cycles of a graph are edge subsets arising from the edges of its cycles by deleting the maximal edge and that their set is denoted by $\mathcal{B}(G, <)$ (Definition 2.31).

The Broken-cycle Theorem was first given by Whitney [151, Theorem 1] and states a combinatorial interpretation of the coefficients of the chromatic polynomial. Originally, the Broken-cycle Theorem was given by removing edges, “ $(-1)^i m_i$ is the number of ways of picking out i arcs from G so that not all the arcs of any broken circuit are removed” [151, Theorem 1]. Later on, the point of view was changed to the nowadays used (and in fact more general) version inserting edges, “the number $(-1)^i m_i$ is the number of subgraphs of G of i arcs which do not contain all the arcs of any broken circuit” [152, Section 7].

We first restate the Broken-cycle Theorem and its proof in order to make the reader familiar with this result and thereby to point to the differences occurring in the following.

Theorem 3.1 (Section 7 in [152], Theorem 2.3.1 in [52]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . The chromatic polynomial $\chi(G, x)$ satisfies*

$$\chi(G, x) = \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}(G, <): B \not\subseteq A}} (-1)^{|A|} x^{k(G \setminus A)} \quad (3.38)$$

$$= \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}(G, <): B \not\subseteq A}} (-1)^{|A|} x^{|V| - |A|}. \quad (3.39)$$

Proof. For each broken cycle $B \in \mathcal{B}(G, <)$, we denote by $e(B)$ the minimal edge closing the broken cycle B , i.e. $e(B) = \min \{e \in E \mid B \cup \{e\} \text{ is the edge set of a cycle in } G\}$. Assume that $\mathcal{B}(G, <) = \{B_1, \dots, B_k\}$, such that $i < j$ if $e(B_i) < e(B_j)$.

We partition the set of edge subsets $A \subseteq E$ into blocks E_i (some of them may be empty), such that $A \in E_i$, if B_i is the minimal broken cycle (with respect to its index and to the edge closing it) included in A , i.e. $A \in E_i$, if $i = \min \{j \mid B_j \subseteq A\}$.

Then for each i and each $A \in E_i$ with $e(B_i) \notin A$, $A \in E_i$ if and only if $A \cup \{e(B_i)\} \in E_i$. For the first direction we assume that $A \cup \{e(B_i)\} \in E_j$ with $i \neq j$. Because $B_i \in A \cup \{e(B_i)\}$, by the definition of E_j it follows that $B_j \in A \cup \{e(B_i)\}$ with $j < i$. But $e(B_i)$ is in the broken cycle B_j , otherwise $A \in B_j$, and therefore $e(B_i) < e(B_j)$. Consequently $i < j$, which gives a contradiction. The second direction follows easily from the fact that if $B_i \subseteq A \cup \{e(B_i)\}$, then $B_i \subseteq A$, and by deleting an edge no other broken cycle occurs.

For such i and A , $e(B_i)$ is an edge of a cycle in $G\langle A \cup \{e(B_i)\} \rangle$. Therefore $k(G\langle A \rangle) = k(G\langle A \cup \{e(B_i)\} \rangle)$ and consequently

$$(-1)^{|A|} x^{k(G\langle A \rangle)} = -(-1)^{|A \cup \{e(B_i)\}|} x^{k(G\langle A \cup \{e(B_i)\} \rangle)}.$$

Hence, for each block $E_i \neq E_0$ it holds

$$\sum_{A \in E_i} (-1)^{|A|} x^{k(G\langle A \rangle)} = 0.$$

As E_0 is the set of edge subsets not including any broken cycle $B \in \mathcal{B}(G, <)$, we have $E_0 = \{A \subseteq E \mid \forall B \in \mathcal{B}(G, <): B \not\subseteq A\}$, and the first statement follows from the edge subset expansion of the chromatic polynomial as given in Equation (3.5):

$$\begin{aligned} \chi(G, x) &= \sum_{A \subseteq E} (-1)^{|A|} x^{k(G\langle A \rangle)} \\ &= \sum_{\substack{A \subseteq E \\ A \in E_0}} (-1)^{|A|} x^{k(G\langle A \rangle)} \\ &= \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}(G, <): B \not\subseteq A}} (-1)^{|A|} x^{k(G\langle A \rangle)}. \end{aligned}$$

Because broken-cycle-free subgraphs are cycle-free subgraphs (forests), they satisfy $k(G\langle A \rangle) = |V| - |A|$ and therefore the second statement holds. \square

The Broken-cycle Theorem as given in Equation (3.38) can be generalized in two aspects:

1. by enabling a restriction of the regarded set of broken cycles,
2. by introducing more general terms of summation.

Theorem 3.2. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E , $\mathcal{B} \subseteq \mathcal{B}(G, <)$ a subset of the set of broken cycles of G , and $f(G, A)$ a function mapping in an additive abelian group, such that for all $A \subseteq E$ and all $e \in E \setminus A$ it holds*

$$k(G\langle A \rangle) = k(G\langle A \cup \{e\} \rangle) \Rightarrow f(G, A) = -f(G, A \cup \{e\}). \quad (3.40)$$

Then

$$\sum_{A \subseteq E} f(G, A) = \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}: B \not\subseteq A}} f(G, A). \quad (3.41)$$

This can be shown by a proof similar to the those for the original statement, where “broken cycle $B \in \mathcal{B}(G, <)$ ” is replaced by “broken cycle $B \in \mathcal{B}$ ” and “ $(-1)^{|A|} x^{k(G(A))}$ ” is replaced by “ $f(G, A)$ ”. We give an alternative proof using induction with respect to the number of broken cycles in \mathcal{B} .

Proof. We use induction with respect to the cardinality of the set \mathcal{B} , that is with respect to the number of broken cycles regarded.

For the basic step we assume that $|\mathcal{B}| = 0$ and the statement holds obviously. We assume as induction hypothesis that the statement holds for any set $\mathcal{B} \subseteq \mathcal{B}(G, <)$ with cardinality less than k and consider now a set $\mathcal{B} \subseteq \mathcal{B}(G, <)$ with cardinality k .

For each broken cycle $B \in \mathcal{B}(G, <)$, we denote by $e(B)$ the maximal edge closing the broken cycle B , i.e. $e(B) = \max \{e \in E \mid B \cup \{e\} \text{ is the edge set of a cycle of } G\}$. Let $B \in \mathcal{B}$ and $\mathcal{B}' = \mathcal{B} \setminus \{B\}$, such that $e(B) \not\prec e(B')$ for all $B' \in \mathcal{B}'$.

In fact, we only have to show that the edge subsets, which do include the broken cycle B , but do not include any broken cycle $B' \in \mathcal{B}'$, cancel each other. Let \mathcal{A} be the set of such edge subsets, i.e.

$$\mathcal{A} = \bigcup_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A \\ B \subseteq A}} \{A\}.$$

Then for each $A \subseteq E$ with $e(B) \notin A$, $A \in \mathcal{A}$ if and only if $A \cup \{e(B)\} \in \mathcal{A}$. For the first direction we assume that $A \cup \{e(B)\}$ includes another broken cycle $B' \in \mathcal{B}'$: Then $e(B)$ must be in the broken cycle B' , but does not close it, otherwise $B' \subseteq A$, and therefore $e(B) < e(B')$. This is a contradiction to $e(B) \not\prec e(B')$ for all $B' \in \mathcal{B}'$, therefore $A \cup \{e(B)\}$ does not include another broken cycle. The second direction follows easily from the fact that if $B \subseteq A \cup \{e(B)\}$, then $B \subseteq A$, and by deleting an edge no new broken cycle can occur.

The statement follows by

$$\begin{aligned} \sum_{A \subseteq E} f(G, A) &= \sum_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A}} f(G, A) \\ &= \sum_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A \\ B \not\subseteq A}} f(G, A) + \sum_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A \\ B \subseteq A}} f(G, A) \\ &= \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}: B \not\subseteq A}} f(G, A) + \sum_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A \\ B \subseteq A, e(B) \in A}} f(G, A) + \sum_{\substack{A \subseteq E \\ \forall B' \in \mathcal{B}': B' \not\subseteq A \\ B \subseteq A, e(B) \notin A}} f(G, A) \end{aligned}$$

$$= \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}: B \not\subseteq A}} f(G, A). \quad \square$$

Furthermore, in many cases $f(G, A) = f(G\langle A \rangle, A)$, that is the function f depends only on the spanning subgraph $G\langle A \rangle$. This is in fact a special case as, for example, the edge set E and therefore the number of edges connecting vertices in different connected components and in the same connected component in $G\langle A \rangle$ is not known, respectively.

For a given linear order $<$ on the edge set E , the conditions given in Equation (3.40) must only be satisfied by the edges closing some broken cycle. Claiming the result for any linear order, the condition is required for any edge which can be an edge of a broken cycle, which are exactly the edges in a cycle.

An inductive proof for the Broken-cycle Theorem with respect to the number of edges is given by Dohmen [48].

Corollary 3.3 (Theorem 9 in [50]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $\mathcal{B} \subseteq \mathcal{B}(G, <)$ a subset of broken cycles of G . The bivariate chromatic polynomial $P(G, x, y)$ satisfies*

$$P(G, x, y) = \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}: B \not\subseteq A}} (-1)^{|A|} x^{i(G\langle A \rangle)} y^{c(G\langle A \rangle)}. \quad (3.42)$$

Proof. The statement follows directly via Theorem 3.2 from the edge subset expansion of the bivariate chromatic polynomial [139, Corollary 29]. \square

Definition 3.4 (Proposition 5.1 in [107]). Let $G = (V, E)$ be a graph. The *U-polynomial* $U(G, X, y)$ is defined as

$$U(G, X, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A \rangle)} (y - 1)^{|A| - |V| + k(G\langle A \rangle)}, \quad (3.43)$$

where $k_i(G)$ is the number of connected components of G with exactly i vertices.

Corollary 3.5. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $\mathcal{B} \subseteq \mathcal{B}(G, <)$ a subset of broken cycles of G . The U-polynomial $U(G, X, y)$ at $y = 0$ satisfies*

$$U(G, X, 0) = \sum_{\substack{A \subseteq E \\ \forall B \in \mathcal{B}: B \not\subseteq A}} \prod_{i=1}^{|V|} x_i^{k_i(G\langle A \rangle)} (-1)^{|A| - |V| + k(G\langle A \rangle)}, \quad (3.44)$$

where $k_i(G)$ is the number of connected components of G with exactly i vertices.

Proof. The statement follows directly via Theorem 3.2 from the definition of the U-polynomial (Definition 3.4). \square

3.2.2 Spanning Forest Representation

The relation between spanning forest representations and edge subset representations follows from the fact that each edge subset arises from exactly one spanning forest by deleting some internally active edges and adding some externally active edges. This was first proven by Crapo [42, Lemma 8] and an analogous result for matroids has been shown by Björner [22, Proposition 7.3.6].

We restate this proof here for various reasons. First, from the results follows a nice relation between the sums over edge subsets and sums over spanning forests (Corollary 3.7). Second, the proof gives a good insight in the consequences of the definition of internally and externally active edges. And third, it can be applied for direct proofs of the edge subset expansion of the Tutte polynomial (Corollary 3.9) and spanning forest expansions of some graph polynomials, including the Potts model (Corollary 3.10) and the U-polynomial (Corollary 3.11).

Please remember the following definitions: The spanning forests of a graph are its inclusion minimal spanning subgraphs with the same number of connected components, and their set is denoted by $\mathcal{F}(G)$ (Definition 2.24). An internally active edge is an edge of the spanning forest, which is the maximal in the cut it crosses (Definition 2.32), and an externally active edge is an edge not in the spanning subgraph, which is the maximal in the cycle it closes (Definition 2.33). The corresponding sets are denoted by $E_i(F, G, <)$ and $E_e(F, G, <)$ and their cardinalities are denoted by $i(F, G, <)$ and $e(F, G, <)$, respectively.

Theorem 3.6 (Lemma 8 in [42]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . Then*

$$\bigcup_{F=(V, A_f) \in \mathcal{F}(G)} [A_f \setminus E_i(F, G, <), A_f \cup E_e(F, G, <)] = \bigcup_{A \subseteq E} \{A\} = 2^E. \quad (3.45)$$

Proof. We have to prove that the intervals for each spanning forest F are mutually disjoint and that each edge subset is in some interval. Let $I(F)$ be the interval arising from the spanning forest F , i.e. $I(F) = [A_f \setminus E_i(F, G, <), A_f \cup E_e(F, G, <)]$ for each $F = (V, A_f) \in \mathcal{F}(G)$.

First, we show that $I(F^1) \cap I(F^2) = \emptyset$ for different spanning forests $F^1, F^2 \in \mathcal{F}(G)$. Assume that there is an edge subset $A \subseteq E$ with $A \in I(F^1) \cap I(F^2)$. As F^1 and F^2 are different spanning forests, there is an edge $g \in E(F^1) \setminus E(F^2)$. Furthermore, for any choice of g , there is an edge $h \in E(F^2) \setminus E(F^1)$, such that $F_{-g+h}^1, F_{-h+g}^2 \in \mathcal{F}(G)$. (There is at least one edge on the path connecting the incident vertices of g in F^2 , which is in the cut crossed by g in F^1 . These conditions ensure that we can “compare” the edges g and h , because g is in the cycle closed by adding h to F^1 and, equivalently, in the cut crossed by h in F^2 , and vice versa.)

We distinguish whether g ($g \in E(F^1)$ but $g \notin E(F^2)$) and h ($h \notin E(F^1)$ but $h \in E(F^2)$) are in A or not:

- Case 1: $g \in A, h \in A$: We have a contradiction by

- $g \in A \Rightarrow g \in E_e(F^2, G, <) \Rightarrow h < g$,
- $h \in A \Rightarrow h \in E_e(F^1, G, <) \Rightarrow g < h$.
- Case 2: $g \in A, h \notin A$: We have a contradiction by
 - $g \in A \Rightarrow g \in E_e(F^2, G, <) \Rightarrow h < g$,
 - $h \notin A \Rightarrow h \in E_i(F^2, G, <) \Rightarrow g < h$.
- Case 3: $g \notin A, h \in A$: We have a contradiction by
 - $g \notin A \Rightarrow g \in E_i(F^1, G, <) \Rightarrow h < g$,
 - $h \in A \Rightarrow h \in E_i(F^1, G, <) \Rightarrow g < h$.
- Case 4: $g \notin A, h \notin A$: We have a contradiction by
 - $g \notin A \Rightarrow g \in E_i(F^1, G, <) \Rightarrow h < g$,
 - $h \notin A \Rightarrow h \in E_i(F^2, G, <) \Rightarrow g < h$.

Hence, there is no such edge subset A and consequently the intervals for different spanning forests are mutually disjoint.

Second, we show that for each edge subset $A \subseteq E$ there is a spanning forest $F \in \mathcal{F}(G)$ with $A \in I(F)$.

We arrange the edges of A and $E \setminus A$ in a sequence $e_1, \dots, e_{|E|}$, such that the edges of A appear before the edges of $E \setminus A$, that the edges of A are increasing, and that the edges of $E \setminus A$ are decreasing, both with respect to $<$.

We start with the edgeless graph on the vertex set V and successively add the edges of E in this graph as they appear in the sequence, but only if the arising graph remains acyclic. That means $G^0 = (V, \emptyset)$ and for $i \in \{1, \dots, |E|\}$ we have

$$G^i = \begin{cases} G_{+e_i}^{i-1} & \text{if } G_{+e_i}^{i-1} \text{ is acyclic,} \\ G^{i-1} & \text{if } G_{+e_i}^{i-1} \text{ is cyclic.} \end{cases}$$

Thus, $G^{|E|} = F = (V, A_f) \in \mathcal{F}(G)$ is a spanning forest of G .

An edge that is in A , but not in A_f , is not added to G^i , meaning that it would close a cycle consisting of earlier added and thus lesser edges of A . Hence, this edge is an externally active edge (maximal edge of the cycles closed by itself), $A \setminus A_f = A_e \subseteq E_e(F, G, <)$.

An edge that is not in A , but in A_f , is added to G^i , meaning that it is the first and thus greatest edges of $E \setminus A$ crossing the according cut. Hence, this edge is an internally active edge (maximal edge of the cut crossed by itself), $A_f \setminus A = A_i \subseteq E_i(F, G, <)$.

Consequently, $(A_f \setminus A_i) \cup A_e = A$, and therefore for each edge subset $A \subseteq E$ there is a spanning forest $F = (V, A_f) \in \mathcal{F}(G)$ such that $A \in I(F)$. \square

Corollary 3.7 (Corollary 7 in [137]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E , $A \subseteq E$ an edge subset of G , and $f(G, A)$ a function in an additive abelian group. Then*

$$\sum_{F \in \mathcal{F}(G)} \sum_{\substack{A = (A_f \setminus A_i) \cup A_e \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} f(G, A) = \sum_{A \subseteq E} f(G, A). \quad (3.46)$$

Proof. The statement follows directly from Theorem 3.6, because for each spanning forest $F \in \mathcal{F}(G)$ we have

$$\bigcup_{\substack{A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} (A_f \setminus A_i) \cup A_e = [A_f \setminus E_i(F, G, <), A_f \cup E_e(F, G, <)],$$

and hence we sum on both sides of the equation over the same edge subsets. \square

When applying the theorems above, it seems useful to point to some kind of disjointness of internally and externally active edges of a spanning forest, which results in some kind of independence of deleting and adding these edges.

Lemma 3.8 (Lemma 9 in [137]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E and $F = (V, A) \in \mathcal{F}(G)$ a spanning forest of G . Then deleting an internally active edge splits a connected component, which can not be reconnected by adding externally active edges. Adding an externally active edge connects vertices connected by a path, which can not be destroyed by deleting internally active edges.*

Proof. The statement follows directly from the definition of internally and externally active edges: Assume there is an internally active edge $e \in A$ and an externally active edge $f \in E \setminus A$, such that the connected components arising by deleting e are connected by adding f or the other way around. Then f is in the cut crossed by e and hence $f < e$ by the definition of internally active edges. But e is in the cycle closed by F and hence $e < f$ by the definition of externally active edges, which gives a contradiction. \square

As announced, we can apply the corollary and lemma above for a direct verification of the edge subset expansion of the Tutte polynomial. Usually, this statement is proven by showing that both the Tutte polynomial and its edge subset expansion satisfy the same recurrence relation and have the same initial value [25, Theorem 10 in Section X.5].

Corollary 3.9 (Equation (9.6.2) in [144]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . The Tutte polynomial $T(G, x, y)$ satisfies*

$$T(G, x, y) = (x - 1)^{-k(G)} (y - 1)^{-|V|} \sum_{A \subseteq E} ((x - 1)(y - 1))^{k(G \setminus A)} (y - 1)^{|A|}. \quad (3.47)$$

Proof. By Corollary 3.7 it holds

$$\begin{aligned}
& (x-1)^{-k(G)}(y-1)^{-|V|} \sum_{A \subseteq E} ((x-1)(y-1))^{k(G\langle A \rangle)} (y-1)^{|A|} \\
&= \sum_{A \subseteq E} (x-1)^{k(G\langle A \rangle)-k(G)} (y-1)^{k(G\langle A \rangle)-|V|+|A|} \\
&= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_f=E(F) \\ A_i \subseteq E_i(F,G,<) \\ A_e \subseteq E_e(F,G,<)}} (x-1)^{k(G\langle A \rangle)-k(G)} (y-1)^{k(G\langle A \rangle)-|V|+|A|}.
\end{aligned}$$

We consider the exponents of $(x-1)$ and $(y-1)$: The exponent of $(x-1)$ is 0 if $A = E(F)$ ($A_i = A_e = \emptyset$), and it increases by 1 with each edge in A_i ($k(G\langle A \rangle)$ increases by 1), while it is not influenced from the edges in A_e . The exponent of $(y-1)$ is also 0 if $A = E(F)$, and it increases by 1 with each edge in A_e ($|A|$ increases by 1), while it is not influenced from the edges in A_i ($k(G\langle A \rangle)$ increases by 1, but $|A|$ decreases by 1). Consequently we have

$$\begin{aligned}
& \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_f=E(F) \\ A_i \subseteq E_i(F,G,<) \\ A_e \subseteq E_e(F,G,<)}} (x-1)^{k(G\langle A \rangle)-k(G)} (y-1)^{k(G\langle A \rangle)-|V|+|A|} \\
&= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_f=E(F) \\ A_i \subseteq E_i(F,G,<) \\ A_e \subseteq E_e(F,G,<)}} (x-1)^{|A_i|} (y-1)^{|A_e|} \\
&= \sum_{F \in \mathcal{F}(G)} x^{i(F,G,<)} y^{e(F,G,<)} \\
&= T(G, x, y). \quad \square
\end{aligned}$$

The following statement, the spanning forest expansion of the Potts model, can alternatively be shown from the definition of the Tutte polynomial by applying the relation between both graph polynomials.

Corollary 3.10. *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . The Potts model $Z(G, x, y)$ satisfies*

$$Z(G, x, y) = \left(\frac{x}{y}\right)^{k(G)} y^{|V|} \sum_{F \in \mathcal{F}(G)} \left(1 + \frac{x}{y}\right)^{i(F,G,<)} (1+y)^{e(F,G,<)}. \quad (3.48)$$

Proof. First, we start with the definition of the Potts model as given in Equation (3.4) and apply Corollary 3.7:

$$Z(G, x, y) = \sum_{A \subseteq E} x^{k(G\langle A \rangle)} y^{|A|}$$

$$= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} x^{k(G \langle A \rangle)} y^{|A|}.$$

Second, we describe the terms for edge subsets corresponding to spanning forests and how these change according to the number of internally and externally active edges: A spanning forest has $k(G)$ connected components and $|V| - k(G)$ edges which generates the monomial $x^{k(G)} y^{|V| - k(G)}$; by deleting an internally active edge the number of connected components increases by 1 and the number of edges decreases by 1 ($\frac{x}{y}$); and by inserting an externally active edge the number of edges increases by 1 (y). Hence we have

$$\begin{aligned} Z(G, x, y) &= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A=(A_f \setminus A_i) \cup A_e \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} x^{k(G)} y^{|V| - k(G)} \left(1 + \frac{x}{y}\right)^{|A_i|} (1 + y)^{|A_e|} \\ &= \left(\frac{x}{y}\right)^{k(G)} y^{|V|} \sum_{F \in \mathcal{F}(G)} \left(1 + \frac{x}{y}\right)^{i(F, G, <)} (1 + y)^{e(F, G, <)}. \quad \square \end{aligned}$$

Along the same line of argumentation the spanning forest expansion of the chromatic polynomial (Equation (3.10)) can be shown [137, Theorem 12]. Alternatively, this can be derived from the Tutte polynomial [141, Equation (4) and (21); 15, Theorem 14.1].

In the results above, we have given spanning forest expansions depending only on the number of internally and externally active edges, but not on the edges itself. The reason is, that these graph polynomials count parameters, whose values for a spanning forest can be determined and whose change for each internally active edge deleted and each externally active edge added can be quantified.

Sometimes, this is only possible for the change arising by one of the two operations. For example, for the U -polynomial only the change arising by the addition of an externally active edge is of such an easy form, because these edges are added between vertices in the same connected component and therefore do not change the distribution of the vertices in the connected components.

Corollary 3.11 (Theorem 17 in [137]). *Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . The U -polynomial $U(G, X, y)$ satisfies*

$$U(G, X, y) = \sum_{F \in \mathcal{F}(G)} y^{e(F, G, <)} \sum_{\substack{A=A_f \setminus A_i \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <)}} x^*(G \langle A \rangle), \quad (3.49)$$

where $x^*(G) = \prod_{i=1}^{|V|} x_i^{k_i(G)}$ and $k_i(G)$ is the number of connected components of G with exactly i vertices.

Proof. Applying Corollary 3.7 to the definition of the U -polynomial we have

$$\begin{aligned} U(G, X, y) &= \sum_{A \subseteq E} x^*(G\langle A \rangle) (y-1)^{|A| - |V| + k(G\langle A \rangle)} \\ &= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A = (A_f \setminus A_i) \cup A_e \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} x^*(G\langle A \rangle) (y-1)^{|A| - |V| + k(G\langle A \rangle)}. \end{aligned}$$

Therein we have to describe the terms for edge sets corresponding to spanning trees and how they change according to the number of internally and externally active edges: The exponent of $(y-1)$ for the edge subset corresponding to an spanning forests equals 0. If an externally active edge is added, then the arrangement of the vertices in the connected components is not affected, but the number of edges increases by one $((y-1))$. If an internally active edge is deleted, then the number of edges decreases by one and the number of connected components increases by one, hence the exponent of $(y-1)$ is not affected. (But the arrangement of the vertices in the connected components changes and does not only depend on the number of internally active edges deleted.) It follows

$$\begin{aligned} U(G, X, y) &= \sum_{F \in \mathcal{F}(G)} \sum_{\substack{A = (A_f \setminus A_i) \cup A_e \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <) \\ A_e \subseteq E_e(F, G, <)}} x^*(G\langle A \rangle) (y-1)^{|A_e|} \\ &= \sum_{F \in \mathcal{F}(G)} y^{e(F, G, <)} \sum_{\substack{A = A_f \setminus A_i \\ A_f = E(F) \\ A_i \subseteq E_i(F, G, <)}} x^*(G\langle A \rangle). \quad \square \end{aligned}$$

3.2.3 Reliability Domination Representation

We have introduced two definitions of signed domination. For the first one, an edge subset expansion follows directly by Möbius inversion (Theorem 2.44). We show that the second one has the same edge subset expansion and therefore both definitions are equivalent to each other.

Please remember the two alternative definitions of signed domination of a graph for an edge subset A and an integer k . Either it can be defined implicitly such that its sum over all subsets of the edge subsets equals 1 or 0, depending on whether the subgraph spanned by the edge subset has at most k connected components or not, and is denoted as $d(G, A, k)$ (Definition 2.43). Or it is defined as the number of k -formations of the subgraph spanned by the edge subset of odd cardinality minus those of even cardinality, and is denoted as $d'(G, A, k)$ (Definition 2.45).

Theorem 3.12 (Proposition 2.8 in [82]). *Let $G = (V, E)$ be a graph, $A \subseteq E$ an edge*

subset of G and $k \in \mathbb{N}$. Then

$$d'(G, A, k) = \sum_{B \subseteq A} (-1)^{|A|-|B|} [k(G\langle B \rangle) \leq k] = d(G, A, k). \quad (3.50)$$

Proof. For each edge subset $B \subseteq E$ of G , the spanning subgraph $G\langle B \rangle$ has at most k connected components, if and only if there is a k -forest of $G\langle B \rangle$, therefore we have

$$\begin{aligned} [k(G\langle B \rangle) \leq k] &= 1 - [\mathcal{F}(G, B, k) = \emptyset] \\ &= 1 - 0^{|\mathcal{F}(G, B, k)|} \\ &= 1 - (1 - 1)^{|\mathcal{F}(G, B, k)|} \\ &= 1 - \sum_{F \subseteq \mathcal{F}(G, B, k)} (-1)^{|F|} \\ &= \sum_{\emptyset \subset F \subseteq \mathcal{F}(G, B, k)} (-1)^{|F|-1} \\ &= \sum_{\substack{A \subseteq B \\ \emptyset \subset F \subseteq \mathcal{F}(G, B, k) \\ \bigcup_{A' \in F} A' = A}} (-1)^{|F|-1}. \end{aligned}$$

By Möbius inversion it follows that

$$\sum_{B \subseteq A} (-1)^{|A|-|B|} [k(G\langle B \rangle) \leq k] = \sum_{\substack{\emptyset \subset F \subseteq \mathcal{F}(G, A, k) \\ \bigcup_{A' \in F} A' = A}} (-1)^{|F|-1}.$$

Then, $d(G, A, k)$ equals the left hand side and the right hand side equals $d'(G, A, k)$ by

$$\begin{aligned} d(G, A, k) &= \sum_{B \subseteq A} (-1)^{|A|-|B|} [k(G\langle B \rangle) \leq k] \\ &= \sum_{\substack{\emptyset \subset F \subseteq \mathcal{F}(G, A, k) \\ \bigcup_{A' \in F} A' = A}} (-1)^{|F|-1} \\ &= \sum_{D \in \mathcal{D}(G, A, k)} (-1)^{|F|-1} \\ &= d'(G, A, k). \quad \square \end{aligned}$$

The theorem above can be applied to show the reliability domination representation of the reliability polynomial and the chromatic polynomial.

Theorem 3.13 (Equation (7) in [124]). *Let $G = (V, E)$ be a graph. The reliability polynomial $R(G, p)$ satisfies*

$$R(G, p) = \sum_{B \subseteq E} d(G, B, 1) p^{|B|}. \quad (3.51)$$

Proof. The statement follows from the definition of the reliability polynomial [149, Equation (3.4.1)] by

$$\begin{aligned}
R(G, p) &= \sum_{A \subseteq E} p^{|A|} (1-p)^{|E|-|A|} [k(G \langle A \rangle) = 1] \\
&= (1-p)^{|E|} \sum_{A \subseteq E} \left(\frac{p}{1-p}\right)^{|A|} [k(G \langle A \rangle) = 1] \\
&= (1-p)^{|E|} \sum_{A \subseteq E} \left(\frac{p}{1-p}\right)^{|A|} \sum_{B \subseteq A} d(G, B, 1) \\
&= (1-p)^{|E|} \sum_{B \subseteq E} \left(\frac{p}{1-p}\right)^{|B|} \left(1 + \frac{p}{1-p}\right)^{|E|-|B|} d(G, B, 1) \\
&= \sum_{B \subseteq E} d(G, B, 1) p^{|B|}. \quad \square
\end{aligned}$$

Theorem 3.14 ([125], due to Rodriguez). *Let $G = (V, E)$ be a graph. The chromatic polynomial satisfies*

$$\chi(G, x) = (-1)^{|E|} (1-x) \sum_{k=1}^{|V|-1} d(G, E, k) x^k. \quad (3.52)$$

Proof. The statement follow from the edge subset expansion of the chromatic polynomial as given in Equation (3.5) by

$$\begin{aligned}
\chi(G, x) &= \sum_{A \subseteq E} (-1)^{|A|} x^{k(G \langle A \rangle)} \\
&= \sum_k \sum_{A \subseteq E} (-1)^{|A|} [k(G \langle A \rangle) = k] x^k \\
&= \sum_k (-1)^{|E|} (d(G, E, k) - d(G, E, k-1)) x^k \\
&= (-1)^{|E|} \left(\sum_k d(G, E, k) x^k - x \sum_k d(G, E, k) x^k \right) \\
&= (-1)^{|E|} (1-x) \sum_{k=1}^{|V|-1} d(G, E, k) x^k,
\end{aligned}$$

where the third identity is by

$$d(G, E, k) - d(G, E, k-1) = \sum_{A \subseteq E} (-1)^{|E|-|A|} [k(G \langle A \rangle) = k]. \quad \square$$

3.3 Recurrence Relations

In this section we delve a little bit more deeply into recurrence relations, either used to define graph polynomials or satisfied by them. We give some more examples of recurrence relations and proof two general theorems concerning them.

3.3.1 Some Examples

The probably first recurrence relation introduced for a graph polynomial was the one for the chromatic polynomial $\chi(G, x)$ of a graph $G = (V, E)$ with an edge $e \in E$:

$$\chi(G, x) = \chi(G_{-e}, x) - \chi(G_{/e}, x). \quad (3.53)$$

According to Kung [92], this equality is due to Foster and was first published by Whitney [153, Note on page 718].

The equality above is often given in a “non-edge-version”: For a (not necessary) two-element vertex subset $f \subseteq V$ with $f \notin E$ it holds [52, Theorem 1.3.1]

$$\chi(G, x) = \chi(G_{+f}, x) + \chi(G_{/f}, x). \quad (3.54)$$

Together with the multiplicativity in components (Equation (3.36)) and the initial value (Equation (3.37)) the chromatic polynomial of every graph can be determined. Alternatively, it is sufficient to have the recurrence relation and the values for the edgeless graphs,

$$\chi(E_n, x) = x^n. \quad (3.55)$$

In fact, an analogous equation holds for every graph polynomial which is multiplicative in components and has the initial value x for a single vertex. In Theorem 3.16 we give conditions under which the reverse holds.

Combining the recurrence relation (Equation (3.53)) and the equality above we can give a single identity for the chromatic polynomial from which, in principle, the value of every graph can be calculated:

$$\chi(G \cup E_n, x) = \chi(G_{-e}, x) \cdot x^n - \chi(G_{/e}, x) \cdot x^n. \quad (3.56)$$

The selection of one of the two merged equalities is wrapped in the choice of the components G and E_n .

While the recurrence relations introduced so far, and also the ones in the remainder, are all linear, there is no such restriction. For example, the characteristic polynomial $\phi(G, x)$ of a simple graph $G = (V, E)$ with an edge $e = \{u, v\} \in E$ satisfies [117, Equation (6)]

$$\begin{aligned} \phi(G, x) &= \phi(G_{-e}, x) - \phi(G_{\ominus u \ominus v}, x) \\ &\quad - 2[\phi(G_{\ominus u}, x)\phi(G_{\ominus v}, x) - \phi(G_{-e}, x)\phi(G_{\ominus u \ominus v}, x)]^{\frac{1}{2}}. \end{aligned} \quad (3.57)$$

For a not necessarily simple graph G without any edges parallel to e , also the equality [119, Theorem 1.3]

$$\begin{aligned} \phi(G, x) &= \phi(G_{-e}, x) + \phi(G_{/e}, x) + (x - 1) \cdot \phi(G_{\dot{+}e}, x) \\ &\quad - \phi(G_{\ominus u}, x) - \phi(G_{\ominus v}, x) \end{aligned} \quad (3.58)$$

holds. This equality looks similar to a recurrence relation we prove in Chapter 5. But in contrast to the situation there and to all other recurrence relations discussed, the formula above can not be applied successively to parallel edges. Instead, all parallel edges must be handled at once: If $F \subseteq E$ is the multiset of all edges connecting u and v , then Rowlinson [119, Theorem 1.3] shows that

$$\begin{aligned} \phi(G, x) &= \phi(G_{-F}, x) + |F| \cdot \phi(G_{/F}, x) + |F|(x - |F|) \cdot \phi(G_{\dagger F}, x) \\ &\quad - |F| \cdot \phi(G_{\ominus u}, x) - |F| \cdot \phi(G_{\ominus v}, x). \end{aligned} \quad (3.59)$$

In general, there is no restriction on the graph operations or on the dependence of the coefficients used to define a graph polynomial. The only requirement is that the arising graph polynomial is a graph invariant, that means its value does not depend on a relabeling of the vertices or on the order of the vertices / edges to which the operations are applied. Proofs checking the “invariance” of a recurrence relation are first given by Averbouch, Godlin and Makowsky [4, Section 3; 3, Section 3.3], we state such a result in Section 5.3.3.

3.3.2 Two General Theorems

A local graph operation (for an edge or a vertex) is an operation which only affects the connected component where the corresponding element belongs to.

Definition 3.15. Let $G = (V, E)$ be a graph and $g \in V \cup E$ an element of G . A *local graph operation* is a graph operation $o(g)$, such that if $G = G^1 \cup G^2$ for two graphs $G^1 = (V^1, E^2)$, G^2 and $g \in V^1 \cup E^1$, then

$$G_{o(g)} = G_{o(g)}^1 \cup G^2, \quad (3.60)$$

where $G_{o(g)}$ is the graph arising from G by applying $o(g)$.

According to the definition above, most of the graph operations mentioned (Section 2.6) are local graph operations, including the deletion, contraction, extraction of an edge and also the deletion and contraction of a vertex. An exception is the NA-Kellman’s operation, the Kellman’s operation for non-adjacent vertices [43, Definition 2.8.1].

Theorem 3.16. Let $P(G) = P(G, X)$ be a graph polynomial satisfying a linear recurrence relation with respect to some local graph operations reducing the number of edges and $P(E_n) = x^n$ for some $x \in \mathbb{R}[X]$. Then the graph polynomial $P(G)$ is multiplicative in components, that is for graphs G^1 and G^2 it holds

$$P(G^1 \cup G^2) = P(G^1) \cdot P(G^2). \quad (3.61)$$

Proof. We assume that for a graph $G = (V, E)$, the graph polynomial $P(G)$ satisfies

$$P(G) = \sum_{i=1}^k a_i \cdot P(G_i),$$

$$P(E_n) = x^n,$$

where $k \in \mathbb{N}$, $x, a_i \in \mathbb{R}[X]$ and the graphs G_i are arising from G by local graph operations reducing the number of edges.

We use induction with respect to $|E|$, the number of edges in $G = G^1 \cup G^2$. As basic step we assume that $|E| = 0$. Then $G^1 \cup G^2$ is an edgeless graph, and G^1 and G^2 are isomorphic to E_r and E_s for some $r, s \in \mathbb{N}$, respectively. Consequently, the statement holds by

$$\begin{aligned} P(G^1 \cup G^2) &= P(E_r \cup E_s) = P(E_{r+s}) = x^{r+s} \\ &= x^r \cdot x^s = P(E_r) \cdot P(E_s) = P(G^1) \cdot P(G^2). \end{aligned}$$

As induction hypothesis we assume that the statement holds for graphs with $|E| < m$ and consider now a graph with $|E| = m$. Then for $G = G^1 \cup G^2$ it holds

$$\begin{aligned} P(G^1 \cup G^2) &= P(G) \\ &= \sum_{i \in I} a_i \cdot P(G_i) \\ &= \sum_{i \in I} a_i \cdot P(G_i^1 \cup G^2) \\ &= \sum_{i \in I} a_i \cdot P(G_i^1) \cdot P(G^2) \\ &= \left[\sum_{i \in I} a_i \cdot P(G_i^1) \right] \cdot P(G^2) \\ &= P(G^1) \cdot P(G^2). \end{aligned} \quad \square$$

Remark 3.17. In the theorem above we restrict the graph operations which may be used to those reducing the number of edges. This is necessary for the inductive proof with respect to the number of edges. However, it seems possible to weaken this condition and to require only that by a repeated application of the recurrence relation we end up with edgeless graphs. (In this case the induction can be done with respect to the maximal number of times the recurrence relation must be applied such that all arising graphs are edgeless.) Furthermore, we have required that the coefficients in the linear recurrence relation are constant. In fact, these coefficients can depend on the connected components where the handled element belongs to. (The used functions must be “locally” in the same sense as the graph operations.)

Theorem 3.18. Let $G = (V, E)$ be a graph, $\{G_i\}_{i \in I}$ a family of graphs and $y, a_i \in \mathbb{R}[X]$ for $i \in I$. Let $p(G)$ be a graph parameter. If a graph polynomial $P(G, X)$ satisfies

$$P(G, X) = \sum_{i \in I} a_i \cdot P(G_i, X), \tag{3.62}$$

and a graph polynomial $P'(G, X)$ satisfies

$$P'(G, X) = y^{p(G)} \cdot P(G, X), \quad (3.63)$$

then

$$P'(G, X) = \sum_{i \in I} y^{p(G) - p(G_i)} \cdot a_i \cdot P'(G_i, X). \quad (3.64)$$

Proof. The statement follows directly from the definition of $P'(G, X)$:

$$\begin{aligned} P'(G, X) &= y^{p(G)} \cdot P(G, X) \\ &= y^{p(G)} \cdot \sum_{i \in I} a_i \cdot P(G_i, X) \\ &= \sum_{i \in I} y^{p(G) - p(G_i)} \cdot a_i \cdot y^{p(G_i)} \cdot P(G_i, X) \\ &= \sum_{i \in I} y^{p(G) - p(G_i)} \cdot a_i \cdot P'(G_i, X). \end{aligned} \quad \square$$

Consider for example as graph parameters the number of vertices and the number of connected components of a graph $G = (V, E)$. Then for a graph polynomial $P(G) = P(G, X)$ and $a, b, c, d \in \mathbb{R}[X]$ with

$$P(G) = a \cdot P(G_{-e}) + b \cdot P(G_{/e}) \quad (3.65)$$

it follows that

$$c^{|V|} \cdot P(G) = a \cdot P(G_{-e}) + bc^{|e|-1} \cdot P(G_{/e}), \quad (3.66)$$

and

$$d^{k(G)} \cdot P(G) = ad^{k(G) - k(G_{-e})} \cdot P(G_{-e}) + b \cdot P(G_{/e}). \quad (3.67)$$

The value $|e| - 1$ in the first equality depends (for graphs) on whether the edge is a link or a loop, the value $k(G) - k(G_{-e})$ in the second equality depends on whether the edge is a bridge or not. This in some sense explains, why on the one hand, the Tutte polynomial and the Potts model are strongly related to each other [129, Equation (2.26)],

$$T(G, x, y) = (x - 1)^{k(G)} (y - 1)^{-|V|} \cdot Z(G, (x - 1)(y - 1), y - 1). \quad (3.68)$$

But on the other hand, only the Potts model satisfies a recurrence relation for an arbitrary edge [4, Equation (7)], the recurrence relation for the Tutte polynomial depends on the kind of the edge (loop, bridge, other edge) to which it is applied [141, Equation (19) and (20); 26, Section X.1].

Chapter 4

Edge Elimination Polynomials

In this chapter we present the edge elimination polynomial and some graph polynomials equivalent to it, which we all together subsume under the term *edge elimination polynomials*. Thereby we relate a definition using recurrence relations to definitions counting subgraphs and counting colorings.

While the edge elimination polynomial generalizes a lot of graph polynomials, each of them containing a lot of combinatorial information, very few is known about (additional) combinatorial information encoded in the edge elimination polynomial itself. The recursive definition of this graph polynomial constrains the direct access to such data, and justifies the seeking for and observation of graph polynomials equivalent to the edge elimination polynomial, but with straightforward combinatorial interpretations.

In Section 4.1 we present

- the edge elimination polynomial [4] (defined by a recurrence relation)

which is the source of all but one of the following graph polynomials. Then we introduce equivalent graph polynomials, namely

- the covered components polynomial [139] (counting spanning subgraphs),
- the subgraph counting polynomial [138] (counting subgraphs),
- the extended subgraph counting polynomial (counting subgraphs),
- the trivariate chromatic polynomial [138] (counting colorings),

in Section 4.2 to 4.5, respectively. In Section 4.6 we mention three more edge elimination polynomials available in the literature, these are

- the hyperedge elimination polynomial [150],
- the subgraph enumerating polynomial [29],
- the trivariate chromatic polynomial by White [150].

At the end of this chapter, in Section 4.7 and Section 4.8, we use appropriate graph polynomials to derive some properties and relations valid for all edge elimination polynomials.

While most of the definitions and statements in this chapter are given for graphs, many of them are also valid in the case of hypergraphs. For this, it may be necessary to generalize known graph polynomials to hypergraphs, their definitions are given accordingly in Appendix A.

4.1 The Edge Elimination Polynomial

Motivated by several graph polynomials satisfying a linear recurrence relation with respect to (two of the) edge operations, deletion, contraction and extraction, Averbouch, Godlin and Makowsky [4; 5] define the edge elimination polynomial of a graph.

Definition 4.1 (Equation (13) in [4]). Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The *edge elimination polynomial* $\xi(G) = \xi(G, x, y, z)$ is defined as

$$\xi(G) = \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{+e}), \quad (4.1)$$

$$\xi(G^1 \cup G^2) = \xi(G^1) \cdot \xi(G^2), \quad (4.2)$$

$$\xi(K_1) = x. \quad (4.3)$$

The authors verify that the edge elimination polynomial is “a most general graph polynomial” [4, Theorem 3] satisfying such “most general recurrence relation” [4, Section 2], both with respect to the given edge operations for an invariant graph polynomial. This is investigated in most detail in the PhD thesis of Averbouch [3, Section 3.3].

Theorem 4.2 (Theorem 3 in [4]). Let $G = (V, E)$, G^1, G^2 be graphs. Each graph polynomial $P(G) = P(G, a, b, c, d)$ satisfying

- a linear recurrence relation with respect to the deletion, contraction and extraction for each edge $e \in E$ of G , i.e.

$$P(G) = a \cdot P(G_{-e}) + b \cdot P(G_{/e}) + c \cdot P(G_{+e}), \quad (4.4)$$

- multiplicativity in components, i.e.

$$P(G^1 \cup G^2) = P(G^1) \cdot P(G^2), \quad (4.5)$$

- and an initial condition, i.e.

$$P(K_1) = d, \quad (4.6)$$

can be calculated from the edge elimination polynomial.

From its definition and the theorem above it follows, that the edge elimination polynomial generalizes all graph polynomials obeying similar recurrence relations and graph polynomials that can be calculated from these. This includes for example the bad coloring polynomial, the bivariate chromatic polynomial, the matching polynomial, the Potts model, the Tutte polynomial, and the vertex-cover polynomial. For some of the exact relations, see [4, Remark 4; 6, Section 2].

A first combinatorial interpretation of the coefficients of the edge elimination polynomial is given in terms of a 3-partition of the edge set.

Theorem 4.3 (Theorem 5 in [4]). *Let $G = (V, E)$ be a graph. The edge elimination polynomial $\xi(G, x, y, z)$ satisfies*

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(G \langle A \cup B \rangle) - c(G \langle B \rangle)} y^{|A| + |B| - c(G \langle B \rangle)} z^{c(G \langle B \rangle)}, \quad (4.7)$$

where $(A \sqcup B) \subseteq E$ is used for the summation over pairs of edge subsets (A, B) : $A, B \subseteq E$, such that the set of vertices incident to the edges of A and B are disjoint: $\bigcup_{e \in A} e \cap \bigcup_{e \in B} e = \emptyset$.

4.2 The Covered Components Polynomial

The covered components polynomial of a graph is the generating function for the number of edges, connected components and covered connected components in its spanning subgraphs.

Definition 4.4 (Definition 3 and Definition 41 in [139]). Let $G = (V, E)$ be a hypergraph. The *covered components polynomial* $C(G, x, y, z)$ is defined as

$$C(G, x, y, z) = \sum_{A \subseteq E} x^{k(G \langle A \rangle)} y^{|A|} z^{c(G \langle A \rangle)}. \quad (4.8)$$

The definition is motivated by two properties of the edge elimination polynomial. First, this graph polynomial generalizes the Potts model, and, second, it has an expansion using the number of covered connected components. The covered components polynomial generalizes the Potts model by additionally counting the number of covered connected components.

Proposition 4.5. *Let $G = (V, E)$ be a graph. The covered components polynomial $C(G, x, y, z)$ generalizes the Potts model $Z(G, x, y)$ by*

$$Z(G, x, y) = C(G, x, y, 1). \quad (4.9)$$

Theorem 4.6 (Theorem 4 in [139]). *Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The covered components polynomial $C(G) = C(G, x, y, z)$ satisfies*

$$C(G) = C(G_{-e}) + y \cdot C(G_{/e}) + (xyz - xy) \cdot C(G_{\dagger e}), \quad (4.10)$$

$$C(G_1 \cup G_2) = C(G_1) \cdot C(G_2), \quad (4.11)$$

$$C(K_1) = x. \quad (4.12)$$

Proof. The second equality (multiplicativity in components) holds as the edge subsets (spanning subgraphs) in different components can be chosen independently from each other and the third equality (initial value) holds by definition. Therefore, it only remains to show the first equality.

Let c_1, c_2, c_3 be the covered components polynomials of G restricted to those edge subsets $A \subseteq E$, such that the edge e is not in A , only e but no edge adjacent to it is in A , and e and at least one adjacent edge is in A , respectively, i.e.

$$\begin{aligned} c_1 &= \sum_{A \subseteq E} [e \notin A] x^{k(G \setminus A)} y^{|A|} z^{c(G \setminus A)}, \\ c_2 &= \sum_{A \subseteq E} [e \in A \wedge \nexists f \in A: e \cap f \neq \emptyset] x^{k(G \setminus A)} y^{|A|} z^{c(G \setminus A)}, \\ c_3 &= \sum_{A \subseteq E} [e \in A \wedge \exists f \in A: e \cap f \neq \emptyset] x^{k(G \setminus A)} y^{|A|} z^{c(G \setminus A)}. \end{aligned}$$

The covered components polynomial of G_{-e} counts exactly those edge subsets A not including e , i.e.

$$C(G_{-e}) = c_1.$$

The covered components polynomial of $G_{/e}$ counts exactly those edge subsets A including e , because contracting e keeps the connection properties. But the polynomial is divided by y , as e is not counted, and, in case no edge adjacent to e is in A is also divided by z , as the single vertex to which e is contracted is not counted as a covered connected component, i.e.

$$C(G_{/e}) = \frac{c_2}{yz} + \frac{c_3}{y}.$$

The covered components polynomial of $G_{\dagger e}$ counts exactly those edge subsets A including neither e nor any edge adjacent to it. Therefore, $xyz \cdot C(G_{\dagger e})$ counts those edge subsets A including e but no edge adjacent to it, where the factor corresponds to the (covered) connected component consisting of e and its adjacent vertices, i.e.

$$C(G_{\dagger e}) = \frac{c_2}{xyz}.$$

Thus, the recurrence relation equals the sum of the three distinct cases:

$$C(G_{-e}) + y \cdot C(G_{/e}) + (xyz - xy) \cdot C(G_{\dagger e}) = c_1 + c_2 + c_3 = C(G). \quad \square$$

In the proof above we start with the subgraphs arising in the recurrence relation and determine which situations these are counting. We also can do the other way around, starting with the three distinct situations and determining by which subgraphs these are enumerated [139, Proof of Theorem 4]. (The same holds for analogous proofs in the following sections.)

From the recurrence relation the equivalence of the covered components polynomial and the edge elimination polynomial follows.

Corollary 4.7 (Theorem 5, Corollary 6 in [139]). *Let $G = (V, E)$ be a graph. The covered components polynomial $C(G, x, y, z)$ and the edge elimination polynomial $\xi(G, x, y, z)$ are equivalent graph polynomials related by*

$$C(G, x, y, z) = \xi(G, x, y, xyz - xy), \quad (4.13)$$

$$\xi(G, x, y, z) = C(G, x, y, \frac{z}{xy} + 1). \quad (4.14)$$

Proof. The first equality follows directly from the theorem above, the second one follows by algebraic transformations. \square

For a direct and more algebraic proof, see [139, Proof of Theorem 5].

4.3 The Subgraph Counting Polynomial

The subgraph counting polynomial of a graph is the generating function for the number of vertices, edges and connected components in its subgraphs.

Definition 4.8 (Definition 2 in [138]). Let $G = (V, E)$ be a hypergraph. The *subgraph counting polynomial* $H(G, v, x, y)$ is defined as

$$H(G, v, x, y) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|}. \quad (4.15)$$

The subgraph counting polynomial generalizes the Potts model by summing over all subgraphs instead only over spanning subgraphs.

Proposition 4.9. *Let $G = (V, E)$ be a graph. The subgraph counting polynomial $H(G, v, x, y)$ generalizes the Potts model $Z(G, x, y)$ by*

$$Z(G, x, y) = [v^{|V|}] (H(G, v, x, y)), \quad (4.16)$$

and has the vertex subset expansion

$$H(G, v, x, y) = \sum_{W \subseteq V} v^{|W|} \cdot Z(G[W], x, y). \quad (4.17)$$

The definition of the subgraph counting polynomial is motivated by the following consideration: The covered components polynomial counts the number of connected components and covered connected components, and therefore, as their difference, also the number of isolated vertices. Deleting a subset of the isolated vertices from the spanning subgraphs, all subgraphs are generated. This idea is applied in the proof of the following theorem.

Theorem 4.10. *Let $G = (V, E)$ be a graph. The subgraph counting polynomial $H(G, v, x, y)$ and the covered components polynomial $C(G, x, y, z)$ are equivalent graph polynomials related by*

$$H(G, v, x, y) = v^{|V|} \cdot C(G, \frac{1+vx}{v}, y, \frac{vx}{1+vx}), \quad (4.18)$$

$$C(G, x, y, z) = (x - xz)^{|V|} \cdot H(G, \frac{1}{x - xz}, xz, y). \quad (4.19)$$

Proof. We only prove the first equality, the second one follows by algebraic transformations. From the definition of the covered components polynomial (Definition 4.4) we get

$$\begin{aligned} C(G, v + x, y, \frac{x}{v + x}) &= \sum_{F \subseteq E} (v + x)^{k(G \langle F \rangle)} y^{|F|} (\frac{x}{v + x})^{c(G \langle F \rangle)} \\ &= \sum_{F \subseteq E} (v + x)^{i(G \langle F \rangle)} x^{c(G \langle F \rangle)} y^{|F|} \\ &= \sum_{F \subseteq E} \sum_{\bar{W} \subseteq I(G \langle F \rangle)} v^{|\bar{W}|} x^{|I(G \langle F \rangle) \setminus \bar{W}|} x^{c(G \langle F \rangle)} y^{|F|} \\ &= \sum_{\bar{W} \subseteq V} \sum_{F \subseteq E(G[V \setminus \bar{W}])} v^{|\bar{W}|} x^{i((V \setminus \bar{W}), F)} x^{c((V \setminus \bar{W}), F)} y^{|F|} \\ &= \sum_{\bar{W} \subseteq V} \sum_{F \subseteq E(G[V \setminus \bar{W}])} v^{|\bar{W}|} x^{k((V \setminus \bar{W}), F)} y^{|F|} \\ &= \sum_{H=(V \setminus \bar{W}, F) \subseteq G} v^{|\bar{W}|} x^{k(H)} y^{|F|}, \end{aligned}$$

where in the fourth identity we change the summation to determine the subgraph: Instead of first selecting the edges included and then the vertices not included, we first select the vertices not included and then the edges included.

Replacing v by v^{-1} and multiplying with $v^{|V|}$ the statement follows:

$$\begin{aligned} v^{|V|} \cdot C(G, \frac{1 + vx}{v}, y, \frac{vx}{1 + vx}) &= v^{|V|} \sum_{H=(V \setminus \bar{W}, F) \subseteq G} v^{-|\bar{W}|} x^{k(H)} y^{|F|} \\ &= \sum_{H=(W, F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|}. \quad \square \end{aligned}$$

Using the theorem above we can derive a recurrence relation satisfied by the subgraph counting polynomial.

Theorem 4.11 (Theorem 3 in [138]). *Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The subgraph counting polynomial $H(G) = H(G, v, x, y)$ satisfies*

$$H(G) = H(G_{-e}) + v^{|e|-1} y \cdot H(G_{/e}) - v^{|e|-1} y \cdot H(G_{\dagger e}), \quad (4.20)$$

$$H(G^1 \cup G^2) = H(G^1) \cdot H(G^2), \quad (4.21)$$

$$H(K_1) = 1 + vx. \quad (4.22)$$

Proof. The second equality (multiplicativity in components) holds as the subgraphs in different components can be chosen independently from each other and the third equality (initial value) holds by definition. Therefore, it only remains to show the first equality.

Let $\tilde{H} = \tilde{H}(G, v, x, y) = v^{-|V|} \cdot H(G, v, x, y) = C(G, \frac{1+vx}{v}, y, \frac{vx}{1+vx})$. From the recurrence relation for the covered components polynomial (Theorem 4.6), we get

$$\tilde{H}(G) = \tilde{H}(G_{-e}) + y \cdot \tilde{H}(G_{/e}) - \frac{y}{v} \cdot \tilde{H}(G_{\dagger e})$$

for an edge $e \in E$. For $H(G) = H(G, v, x, y)$, it follows

$$\begin{aligned} H(G) &= v^{|V|} \cdot \tilde{H}(G) \\ &= v^{|V|} \cdot \left[\tilde{H}(G_{-e}) + y \cdot \tilde{H}(G_{/e}) - \frac{y}{v} \cdot \tilde{H}(G_{\dagger e}) \right] \\ &= v^{|V|} \cdot \tilde{H}(G_{-e}) + v^{|V|} y \cdot \tilde{H}(G_{/e}) - v^{|V|} \frac{y}{v} \cdot \tilde{H}(G_{\dagger e}) \\ &= v^{|V(G_{-e})|} \cdot \tilde{H}(G_{-e}) + v^{|e|-1} y v^{|V(G_{/e})|} \cdot \tilde{H}(G_{/e}) \\ &\quad - v^{|e|-1} y v^{|V(G_{\dagger e})|} \cdot \tilde{H}(G_{\dagger e}) \\ &= H(G_{-e}) + v^{|e|-1} y \cdot H(G_{/e}) - v^{|e|-1} y \cdot H(G_{\dagger e}). \end{aligned} \quad \square$$

For a direct and more combinatorial proof, see [138, Proof of Theorem 3].

Corollary 4.12 (Corollary 4 in [138]). *Let $G = (V, E)$ be a graph. The subgraph counting polynomial $H(G, v, x, y)$ and the edge elimination polynomial $\xi(G, x, y, z)$ are equivalent graph polynomials related by*

$$H(G, v, x, y) = v^{|V|} \cdot \xi(G, \frac{1+vx}{v}, y, -\frac{y}{v}), \quad (4.23)$$

$$\xi(G, x, y, z) = (x-y)^{|V|} \cdot H(G, \frac{1}{x-y}, y, \frac{z}{x-y}). \quad (4.24)$$

Proof. The statements can be derived via the relations to the covered components polynomial (Corollary 4.7 and Theorem 4.10). \square

Alternatively, the corollary above can be derived directly from the recurrence relation, see [138, Proof of Corollary 4].

4.4 The Extended Subgraph Counting Polynomial

The extended subgraph counting polynomial of a graph is the generating function for the number of vertices, edges, connected components and covered connected components in its subgraphs.

Definition 4.13. Let $G = (V, E)$ be a hypergraph. The *extended subgraph counting polynomial* $H'(G, v, x, y, z)$ is defined as

$$H'(G, v, x, y, z) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|} z^{c(H)}. \quad (4.25)$$

The extension of the subgraph counting polynomial is motivated by the question, whether “the combination” of the covered components polynomial and the subgraph counting polynomial creates a proper generalization or not.

Proposition 4.14. *Let $G = (V, E)$ be a graph. The extended subgraph counting polynomial $H'(G, v, x, y, z)$ generalizes the covered components polynomial $C(G, x, y, z)$ and the subgraph counting polynomial $H(G, v, x, y)$ by*

$$C(G, x, y, z) = [v^{|V|}](H'(G, v, x, y, z)), \quad (4.26)$$

$$H(G, v, x, y) = H'(G, v, x, y, 1). \quad (4.27)$$

It turns out, that the relation (and its proof) between the covered components polynomial and the subgraph counting polynomial (Theorem 4.10) can be easily adapted to the extended subgraph counting polynomial. Consequently, the extended subgraph counting polynomial is not a proper generalization of the others but equivalent to them and thus also to the edge elimination polynomial.

Theorem 4.15. *Let $G = (V, E)$ be a graph. The extended subgraph counting polynomial $H'(G, v, x, y, z)$ and the covered components polynomial $C(G, x, y, z)$ are related by*

$$H'(G, v, x, y, z) = v^{|V|} \cdot C(G, \frac{1+vx}{v}, y, \frac{vxz}{1+vx}), \quad (4.28)$$

$$C(G, x, y, z) = (x - xz)^{|V|} \cdot H'(G, \frac{1}{x - xz}, xz, y, 1). \quad (4.29)$$

Proof. The proof is analogous to the proof of Theorem 4.10. \square

Theorem 4.16. *Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The extended subgraph counting polynomial $H'(G) = H'(G, v, x, y, z)$ satisfies*

$$\begin{aligned} H'(G) &= H'(G_{-e}) + v^{|e|-1}y \cdot H'(G_{/e}) \\ &\quad + v^{|e|-1}y(vxz - vx - 1) \cdot H'(G_{+e}), \end{aligned} \quad (4.30)$$

$$H'(G^1 \cup G^2) = H'(G^1) \cdot H'(G^2), \quad (4.31)$$

$$H'(K_1) = 1 + vx. \quad (4.32)$$

Proof. The proof is analogous to the proof of Theorem 4.11. \square

Corollary 4.17. *Let $G = (V, E)$ be a graph. The extended subgraph counting polynomial $H(G, v, x, y)$ and the edge elimination polynomial $\xi(G, x, y, z)$ are equivalent graph polynomials related by*

$$H'(G, v, x, y, z) = v^{|V|} \cdot \xi(G, \frac{1+vx}{v}, y, xyz - xy - \frac{y}{v}), \quad (4.33)$$

$$\xi(G, x, y, z) = (x - y)^{|V|} \cdot H'(G, \frac{1}{x - y}, y, \frac{z}{x - y}, 1). \quad (4.34)$$

Proof. The statements can be derived via the relations to the covered components polynomial (Corollary 4.7 and Theorem 4.15). \square

Corollary 4.18. *Let $G = (V, E)$ be a graph. The extended subgraph counting polynomial $H'(G, v, x, y, z)$ and the subgraph counting polynomial $H(G, v, x, y)$ are equivalent graph polynomials related by*

$$H'(G, v, x, y, z) = (1 + vx - vxz)^{|V|} \cdot H(G, \frac{v}{1 + vx - vxz}, xz, y) \quad (4.35)$$

$$= (1 + vx - vxz)^{|V|} \cdot H'(G, \frac{v}{1 + vx - vxz}, xz, y, 1), \quad (4.36)$$

$$H(G, v, x, y) = H'(G, v, x, y, 1). \quad (4.37)$$

Proof. The first equality follows by inserting both equalities of one of the relations stated above into each other. The second equality follows from the definition. \square

4.5 The Trivariate Chromatic Polynomial

The trivariate chromatic polynomial¹ of a graph is the generating function for the number of so-called “bad monochromatic” edges in its colorings.

Definition 4.19 (Definition 5 in [138]). Let $G = (V, E)$ be a hypergraph. The *trivariate chromatic polynomial* $\tilde{P}(G, x, y, z)$ is defined (for $x, y \in \mathbb{N}, x \geq y$) as

$$\tilde{P}(G, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \leq y \forall v \in e: \phi(v)=c}} z. \quad (4.38)$$

The well-known chromatic polynomial counts the number of (vertex) colorings not including any monochromatic edges, which are edges whose incident vertices are all colored by the same color. The bad coloring polynomial counts all colorings, but with respect to number of monochromatic edges (also known as “bad edges”), and thereby generalizes the chromatic polynomial. Another generalization of the chromatic polynomial is the bivariate chromatic polynomial, where two color classes are considered, a set of “proper” colors, generating “bad monochromatic” edges, and a set of “arbitrary” colors, generating “good monochromatic” edges. The bivariate chromatic polynomial counts the number of colorings not including any bad monochromatic edge.

The trivariate chromatic polynomial combines both generalizations by counting all colorings using the colors $1, \dots, x$, but with respect to the number of bad monochromatic edges, which are edges whose incident vertices are all colored by the same (proper) color from $1, \dots, y$.

¹The present author has introduced this graph polynomial under the name “bivariate bad coloring polynomial” in several talks, first time at a conference at the Zhejiang Normal University (Jinhua, China) in 2010. Because of the conflict of a “bivariate” polynomial in three variables, the name is changed into the same used by White [150] for an almost similar graph polynomial.

Proposition 4.20. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ generalizes the chromatic polynomial $\chi(G, x)$, the bad coloring polynomial $\tilde{\chi}(G, x, z)$ and the bivariate chromatic polynomial $P(G, x, y)$ by*

$$\chi(G, x) = \tilde{P}(G, x, x, 0), \quad (4.39)$$

$$\tilde{\chi}(G, x, z) = \tilde{P}(G, x, x, z), \quad (4.40)$$

$$P(G, x, y) = \tilde{P}(G, x, y, 0). \quad (4.41)$$

To get a coloring counted by the trivariate chromatic polynomial we can first select a set of vertices which we color (independently) by one of the $x-y$ arbitrary colors, and then color the remaining vertices with one of the y proper colors.

Proposition 4.21. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ satisfies*

$$\tilde{P}(G, x, y, z) = \sum_{W \subseteq V} (x - y)^{|W|} \cdot \tilde{\chi}(G_{\ominus W}, y, z). \quad (4.42)$$

Theorem 4.22 (Theorem 6 in [138]). *Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The trivariate chromatic polynomial $\tilde{P}(G) = \tilde{P}(G, x, y, z)$ satisfies*

$$\tilde{P}(G) = \tilde{P}(G_{-e}) + (z - 1) \cdot \tilde{P}(G_{/e}) + (1 - z)(x - y) \cdot \tilde{P}(G_{+e}), \quad (4.43)$$

$$\tilde{P}(G^1 \cup G^2) = \tilde{P}(G^1) \cdot \tilde{P}(G^2), \quad (4.44)$$

$$\tilde{P}(K_1) = x. \quad (4.45)$$

Proof. We only prove the first equality, the other two follow from the definition and are in full analogy to the chromatic polynomial and the mentioned generalizations.

For the coloring of the vertices incident to the edge e there are three distinct cases:

1. e is not monochromatic, i.e. not all vertices of e are mapped to the same color c : $\nexists c \forall v \in e: \phi(v) = c$,
2. e is bad monochromatic, i.e. all vertices of e are mapped to the same color $c \leq y$: $\exists c \leq y \forall v \in e: \phi(v) = c$,
3. e is good monochromatic, i.e. all vertices of e are mapped to the same color $c > y$: $\exists c > y \forall v \in e: \phi(v) = c$.

Let p_1, p_2 and p_3 be the trivariate chromatic polynomial of G enumerating exactly those colorings of G corresponding to the first, second and third case, respectively. Obviously, $\tilde{P}(G) = p_1 + p_2 + p_3$.

Each coloring of G_{-e} corresponds to a coloring of G , where the number of bad monochromatic edges is counted correctly, except in the second case, where

the vertices incident to e are colored by the same color $c \leq y$ and e is not counted as bad monochromatic, as it does not appear in the graph:

$$\tilde{P}(G_{-e}) = p_1 + \frac{p_2}{z} + p_3.$$

Each coloring of $G_{/e}$ corresponds to a coloring of G , where all vertices incident to e are mapped to the color c , to which the vertex arising through the contraction of e is mapped. This covers the second and third case. But again, in the second case e is not counted as bad monochromatic:

$$\tilde{P}(G_{/e}) = \frac{p_2}{z} + p_3.$$

Each coloring of $G_{\dagger e}$ corresponds to a colorings of G excluding the vertices incident to e . If we assume that the vertices of e , as in the third case, are all colored by the same color $c > y$, then there are $x - y$ for them:

$$\tilde{P}(G_{\dagger e}) = \frac{p_3}{x - y}.$$

The statement follows by

$$\begin{aligned} \tilde{P}(G_{-e}) + (z - 1) \cdot \tilde{P}(G_{/e}) + (1 - z)(x - y) \cdot \tilde{P}(G_{\dagger e}) &= p_1 + p_2 + p_3 \\ &= \tilde{P}(G). \end{aligned} \quad \square$$

From the recurrence relation above it follows, that the trivariate chromatic polynomial is equivalent to the edge elimination polynomial.

Corollary 4.23 (Corollary 7 in [138]). *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the edge elimination polynomial $\xi(G, x, y, z)$ are equivalent graph polynomials related by*

$$\tilde{P}(G, x, y, z) = \xi(G, x, z - 1, (1 - z)(x - y)), \quad (4.46)$$

$$\xi(G, x, y, z) = \tilde{P}(G, x, x + \frac{z}{y}, y + 1). \quad (4.47)$$

Proof. The first equality follows directly from the recurrence relation stated in the theorem above. The second equality follows by algebraic transformations. \square

The following statement can be derived via the corollary above, but we give an independent, more algebraic proof. In fact, we show the edge subset expansion of the trivariate chromatic polynomial by an argumentation analogous to the one used for the edge subset expansion of the bad coloring polynomial [54, Theorem 9.6.6].

Theorem 4.24. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the covered components polynomial $C(G, x, y, z)$ are related by*

$$\tilde{P}(G, x, y, z) = C(G, x, z - 1, \frac{y}{x}), \quad (4.48)$$

$$C(G, x, y, z) = \tilde{P}(G, x, xz, y + 1). \quad (4.49)$$

Proof. We only prove the first equality, the second one follows by algebraic transformations. Starting with the definition of the trivariate chromatic polynomial (Definition 4.4) we get

$$\begin{aligned}
\tilde{P}(G, x, y, z) &= \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \leq y \forall v \in e: \phi(v)=c}} z \\
&= \sum_{\phi: V \rightarrow \{1, \dots, x\}} z^{|\{e \in E \mid \forall v \in e: \phi(v)=c \leq y\}|} \\
&= \sum_{\phi: V \rightarrow \{1, \dots, x\}} (z - 1 + 1)^{|\{e \in E \mid \forall v \in e: \phi(v)=c \leq y\}|} \\
&= \sum_{\phi: V \rightarrow \{1, \dots, x\}} \sum_{A \subseteq \{e \in E \mid \forall v \in e: \phi(v)=c \leq y\}} (z - 1)^{|A|} \\
&= \sum_{A \subseteq E} \sum_{\substack{\phi: V \rightarrow \{1, \dots, x\} \\ A \subseteq \{e \in E \mid \forall v \in e: \phi(v)=c \leq y\}}} (z - 1)^{|A|} \\
&= \sum_{A \subseteq E} x^{i(G \langle A \rangle)} y^{c(G \langle A \rangle)} (z - 1)^{|A|} \\
&= \sum_{A \subseteq E} x^{k(G \langle A \rangle)} (z - 1)^{|A|} \left(\frac{y}{x}\right)^{c(G \langle A \rangle)} \\
&= C(G, x, z - 1, \frac{y}{x}),
\end{aligned}$$

whereby the last but two identity holds as the vertices of each edge in A have to be colored with the same color $c \leq y$. Thus, for the vertices of each covered connected component there are y colors possible and for the isolated vertices there are x colors possible, which can be chosen independently of each other. \square

Corollary 4.25. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ has the edge subset expansion*

$$\tilde{P}(G, x, y, z) = \sum_{A \subseteq E} x^{i(G \langle A \rangle)} y^{c(G \langle A \rangle)} (z - 1)^{|A|}. \quad (4.50)$$

Proof. The proof is included in the proof of the theorem above. \square

Corollary 4.26. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the subgraph counting polynomial are equivalent graph polynomials related by*

$$\tilde{P}(G, x, y, z) = (x - z + 1)^{|V|} \cdot H(G, \frac{1}{x - z + 1}, z - 1, \frac{(1 - z)(x - y)}{x - z + 1}), \quad (4.51)$$

$$H(G, v, x, y) = v^{|V|} \cdot \tilde{P}(G, \frac{1 + vx}{v}, x, y + 1). \quad (4.52)$$

Proof. The statements can be derived via the relations to the edge elimination polynomial (Corollary 4.12 and Corollary 4.23). \square

The trivariate chromatic polynomial and its recurrence relation given above are utilized by Garijo, Goodall and Nešetřil [59, Theorem 34] to prove that the edge elimination polynomial can be stated as counting graph homomorphisms.

Theorem 4.27 (Theorem 34 in [59]). *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ equals (for $x, y, z \in \mathbb{N}, x \geq y$) the number of homomorphisms of G into $K_{x-y}^1 + K_y^z$, where K_n^z is the complete graph of n vertices with z loops attached at each vertex and the graph $K_{x-y}^1 + K_y^z$ arises from the join of a K_{x-y}^1 and a K_y^z ,*

$$\tilde{P}(G, x, y, z) = \text{hom}(G, K_{x-y}^1 + K_y^z). \quad (4.53)$$

4.6 Further Edge Elimination Polynomials

In this section we mention some other graph polynomials equivalent to the edge elimination polynomial, which are given in the literature.

4.6.1 The Hyperedge Elimination Polynomial

The hyperedge elimination polynomial $\xi(G, x, y, z)$ has been introduced by White [150, Section 4] as an (explicit) “hyperedge version” of the edge elimination polynomial. It is defined [150, Definition 1] by an identity similar to the expansion of the edge elimination polynomial in terms of 3-partitions of the edge set [4, Theorem 5] (Theorem 4.3), which instead of the number of covered connected components of a spanning subgraph uses the number of connected components in the edge-induced subgraph.

Definition 4.28 (Definition 1 in [150]). Let $G = (V, E)$ be a hypergraph. The *hyperedge elimination polynomial* $\xi(G, x, y, z)$ is defined as

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(G \langle A \cup B \rangle) - k(G[B])} y^{|A| + |B| - k(G[B])} z^{k(G[B])}, \quad (4.54)$$

where $(A \sqcup B) \subseteq E$ is used for the summation over pairs of edge subsets $(A, B): A, B \subseteq E$, such that the set of vertices incident to the edges of A and B are disjoint: $\bigcup_{e \in A} e \cap \bigcup_{e \in B} e = \emptyset$.

White [150, Proposition 1] shows that the hyperedge elimination polynomial satisfies exactly the same recurrence relation that are used to define the edge elimination polynomial (Definition 4.1).

4.6.2 The Subgraph Enumerating Polynomial

The first graph polynomial defined that is equivalent to the edge elimination polynomial, is in fact not the edge elimination polynomial itself, but the subgraph enumerating polynomial defined by Borzacchini and Pulito more than two decades ago.

Definition 4.29 (Equation (1) in [29]). Let $G = (V, E)$ be a graph. The *subgraph enumerating polynomial* $P(G, u, v, p)$ is defined as

$$P(G, u, v, p) = \sum_{A \subseteq E} u^{\bar{i}(G \langle A \rangle)} v^{|A|} p^{k(G \langle A \rangle)}, \quad (4.55)$$

where $\bar{i}(G)$ is the number of non-isolated vertices in G .

Furthermore, the authors also proved a recurrence relation with respect to the usual three edge elimination operations.

Theorem 4.30 (Theorem 2 in [29]). Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The subgraph enumerating polynomial $P(G) = P(G, u, v, p)$ satisfies

$$P(G) = P(G_{-e}) + u^{|e|-1} v \cdot P(G_{/e}) + u^{|e|-1} (u-1) v p \cdot P(G_{\dagger e}), \quad (4.56)$$

$$P(G^1 \cup G^2) = P(G^1) \cdot P(G^2), \quad (4.57)$$

$$P(K_1) = p. \quad (4.58)$$

The subgraph enumerating polynomial is strongly related to the covered components polynomial, both differ only in the usage of the number of non-isolated vertices and covered connected components, respectively.

Theorem 4.31. Let $G = (V, E)$ be a graph. The subgraph enumerating polynomial $P(G, u, v, p)$ and the covered components polynomial $C(G, x, y, z)$ are equivalent graph polynomials related by

$$P(G, u, v, p) = u^{|V|} \cdot C(G, \frac{p}{u}, v, u), \quad (4.59)$$

$$C(G, x, y, z) = z^{-|V|} \cdot P(G, z, y, xz). \quad (4.60)$$

Proof. The first equality follows from

$$\begin{aligned} P(G, u, v, p) &= \sum_{A \subseteq E} u^{\bar{i}(G \langle A \rangle)} v^{|A|} p^{k(G \langle A \rangle)} \\ &= \sum_{A \subseteq E} u^{|V| - k(G \langle A \rangle) + c(G \langle A \rangle)} v^{|A|} p^{k(G \langle A \rangle)} \\ &= u^{|V|} \cdot \sum_{A \subseteq E} \left(\frac{p}{u}\right)^{k(G \langle A \rangle)} v^{|A|} u^{c(G \langle A \rangle)} \\ &= u^{|V|} \cdot C(G, \frac{p}{u}, v, u), \end{aligned}$$

and the second one by algebraic transformations. \square

4.6.3 The Trivariate Chromatic Polynomial by White

The trivariate chromatic polynomial $P(G, p, q, t)$ as introduced by White [150, Section 6] equals the trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ except a change of the first two variables.

Proposition 4.32. *Let $G = (V, E)$ be a graph. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the trivariate chromatic polynomial by White $P(G, p, q, t)$ are equivalent graph polynomials related by*

$$\tilde{P}(G, x, y, z) = P(G, y, x, z), \quad (4.61)$$

$$P(G, p, q, t) = \tilde{P}(G, q, p, t). \quad (4.62)$$

White [150, Proposition 5] states an vertex subset expansion in terms of the bivariate chromatic polynomial similar to the one for the trivariate chromatic polynomial (Proposition 4.21).

4.7 Properties

In this section we list a few properties for the edge elimination polynomials. Some more results are already known and given in terms of the covered components polynomial [139].

4.7.1 Encoded Invariants

It is known that the covered components polynomial of a graph encodes several graph invariants [139, Section 3]. We restate only two of the results here, mention that a lot of “chromatic invariants” can be derived, and unify two results concerning the number of vertices of a given degree.

That the number of vertices of a graph is encoded in its edge elimination polynomials has been implicitly stated in the results about the equivalence of these graph polynomials. This number can be determined as the degree of the “first” variable of the graph polynomials.

Proposition 4.33. *Let $G = (V, E)$ be a graph. The number of vertices of G is encoded in the edge elimination polynomials:*

$$|V| = \deg_x(\xi(G, x, y, z)), \quad (4.63)$$

$$= \deg_x(C(G, x, y, z)), \quad (4.64)$$

$$= \deg_v(H(G, v, x, y)), \quad (4.65)$$

$$= \deg_v(H'(G, v, x, y, z)), \quad (4.66)$$

$$= \deg_x(\tilde{P}(G, x, y, z)). \quad (4.67)$$

One graph invariant that is not included in many graph polynomials is the number of edge-induced subgraphs with given number of vertices, edges and connected components. This number can be derived from the covered components polynomial [139, Theorem 16], but a generalization can be easily obtained from the coefficients of the extended subgraph counting polynomial.

Proposition 4.34. *Let $G = (V, E)$ be a graph. The number of subgraphs with exactly n vertices, m edges, k connected components, c of them covered connected components, denoted by $g(n, m, k, c)$, is given as a coefficient of the extended subgraph counting polynomial $H'(G, v, x, y, z)$:*

$$g(n, m, k, c) = [v^n x^k y^m z^c](H'(G, v, x, y, z)). \quad (4.68)$$

From the trivariate chromatic polynomial many chromatic invariants can be determined. Beside the usual chromatic number, that is the minimal number of colors necessary for a proper coloring of the vertices, also some extensions up to the number of colors necessary for a partial coloring with a given number of monochromatic edges.

Proposition 4.35. *Let $G = (V, E)$ be a graph. The minimal number of colors necessary for a coloring of all but a vertices such that b edges are monochromatic, denoted by $\chi_{a,b}(G)$, is encoded in the trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$:*

$$\chi_{a,b}(G) = \min \{x \in \mathbb{N} \mid [v^a z^b](\tilde{P}(v+x, x, z)) > 0\}. \quad (4.69)$$

There are already two results concerning the encoding of the number of vertices with a given degree: First, the number of vertices of degree 0, 1 and of minimum degree can be determined from the covered components polynomial [139, Section 3]. Second, the number of vertices of a given degree in a forest is encoded in the edge elimination polynomial [139, Theorem 36] (shown by a proof using the recurrence relation). Using the trivariate chromatic polynomial both results can be unified.

Theorem 4.36 (Theorem 11 in [138]). *Let $G = (V, E)$ be a graph. The number of vertices with degree i in G , $\deg^{-1}(G, i)$, is encoded in the trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$:*

$$\deg^{-1}(G, i) = [v^1 z^{|E|-i}](\tilde{P}(G, v+1, 1, z)), \quad (4.70)$$

where $|E| = \deg_z(\tilde{P}(G, v+1, 1, z))$.

Informal this can be shown as follows: We consider the number of bad monochromatic edges in a coloring using v arbitrary and 1 proper color. Each term including v^1 corresponds to colorings where exactly one vertex is colored by one of the v arbitrary colors and all other vertices are colored by the same proper color. Hence, all edges except the edges incident to the one arbitrary colored vertex are bad monochromatic, and their number is counted in the variable z .

For a proof following this argumentation, see [138, Theorem 11]. Here we give a slightly different proof using the edge subset expansion of the trivariate chromatic polynomial.

Proof. Applying the edge subset expansion of the trivariate chromatic polynomial (Corollary 4.25),

$$\tilde{P}(G, x, y, z) = \sum_{A \subseteq E} x^{i(G \setminus A)} y^{c(G \setminus A)} (z-1)^{|A|},$$

for $\tilde{P}(G, v+1, 1, z)$, it follows

$$\begin{aligned} \tilde{P}(G, v+1, 1, z) &= \sum_{A \subseteq E} (v+1)^{i(G \setminus A)} 1^{c(G \setminus A)} (z-1)^{|A|} \\ &= \sum_{A \subseteq E} (v+1)^{|I(G \setminus A)|} (z-1)^{|A|} \\ &= \sum_{A \subseteq E} \sum_{W \subseteq I(G \setminus A)} v^{|W|} (z-1)^{|A|} \\ &= \sum_{W \subseteq V} \sum_{A \subseteq E(G_{\ominus W})} v^{|W|} (z-1)^{|A|} \\ &= \sum_{W \subseteq V} v^{|W|} \sum_{A \subseteq E(G_{\ominus W})} (z-1)^{|A|} \\ &= \sum_{W \subseteq V} v^{|W|} z^{|E(G_{\ominus W})|}. \end{aligned}$$

Consequently, the coefficient in front of the monomial $v^1 z^{|E|-i}$ counts the number of vertices, whose deletion removes i edges, and hence the number of vertices with degree i in G , $\deg^{-1}(G, i)$. \square

4.7.2 Derivatives

The following theorem is analogous to a statement given for the subgraph enumerating polynomial [29, Theorem 4].

Theorem 4.37. *Let $G = (V, E)$ be a graph. The subgraph counting polynomial $H(G, v, x, y)$ satisfies*

$$|V| \cdot H(G, v, x, y) = \sum_{v \in V} H(G_{\ominus v}, v, x, y) + v \frac{\partial H(G, v, x, y)}{\partial v}. \quad (4.71)$$

Proof. The subgraph counting polynomial $H(G, v, x, y)$ enumerates the number of vertices, edges and connected components in the subgraphs of G , i.e.

$$H(G, v, x, y) = \sum_{i,j,k} h(G, i, j, k) v^i x^j y^k,$$

where $h(G, i, j, k)$ is the number of subgraphs of G with i vertices, j edges and k connected components.

By Kelly's Lemma [85], each subgraph with exactly i vertices is a subgraph of $|V| - i$ vertex-deleted subgraphs (with one of the $|V| - i$ missing vertices deleted), i.e. for all i, j, k we have

$$(|V| - i) \cdot h(G, i, j, k) = \sum_{v \in V} h(G_{\ominus v}, i, j, k).$$

Hence,

$$(|V| - i) \sum_{i,j,k} h(G, i, j, k) v^i x^j y^k = \sum_{v \in V} \sum_{i,j,k} h(G_{\ominus v}, i, j, k) v^i x^j y^k.$$

Using the definition of the subgraph counting polynomial and the identity

$$i \sum_{i,j,k} h(G, i, j, k) v^i x^j y^k = v \frac{\partial \sum_{i,j,k} h(G, i, j, k) v^i x^j y^k}{\partial v},$$

the statement follows:

$$|V| \cdot H(G, v, x, y) = \sum_{v \in V} H(G_{\ominus v}, v, x, y) + v \frac{\partial H(G, v, x, y)}{\partial v}. \quad \square$$

4.7.3 Reconstructability

Kotek [89, Theorem 2.5] has shown that the edge elimination polynomial of a simple graph with at least three vertices is reconstructable from the isomorphism classes of its deck.

This is also possible without having exactly (the isomorphism classes of) the graphs in the deck. It is enough to know the corresponding *incidence matrix* $\mathcal{N}(G)$, defined by Thatte [133], which represents how often any induced subgraph of G is induced in any other.

Corollary 4.38. *Let $G = (V, E)$ be a simple graph with at least three vertices. The covered components polynomial $C(G, x, y, z)$ is reconstructable from the incidence matrix $\mathcal{N}(G)$.*

Proof. The statement follows directly from [133, Lemma 3.14] by the same argument used for the rank polynomial in [133, Lemma 3.15]: From $\mathcal{N}(G)$ the “number of subgraphs with v vertices (none of which isolated), e edges and l [connected] components”, and hence all coefficients of the covered components polynomial can be determined. \square

For the polynomial reconstructability, the probably first affirmative statement in this direction is given by Tutte for the rank polynomial [142; 143].

Definition 4.39 (Equation (5) in [142]). Let $G = (V, E)$ be a graph. The *rank polynomial* $R(G, x, y)$ is defined as

$$R(G, x, y) = \sum_{A \subseteq E} x^{r(G\langle A \rangle)} y^{|A| - r(G\langle A \rangle)}, \quad (4.72)$$

where $r(G\langle A \rangle) = |V| - k(G\langle A \rangle)$.

From this definition it follows, that additionally knowing the number of vertices (or the number of connected components), the rank polynomial is equivalent to the Potts model, and consequently this is also reconstructable from its polynomial deck.

Proposition 4.40. *Let $G = (V, E)$ be a graph. The rank polynomial $R(G, x, y)$ and the Potts model $Z(G, x, y)$ are related by*

$$R(G, x, y) = \left(\frac{x}{y}\right)^{|V|} \cdot Z(G, \frac{y}{x}, y), \quad (4.73)$$

$$Z(G, x, y) = x^{|V|} \cdot R(G, \frac{y}{x}, y). \quad (4.74)$$

Lemma 4.41. *Let $G = (V, E)$ be a simple graph with at least three vertices. The Potts model $Z(G, x, y)$ of G is reconstructable from the polynomial deck $\mathcal{D}_Z(G)$.*

Proof. From the polynomial deck of G for the Potts model, $\mathcal{D}_Z(G)$, the polynomial deck of G for the rank polynomial, $\mathcal{D}_R(G)$, can be calculated. From this the rank polynomial $R(G, x, y)$ can be reconstructed [143, Theorem 7.4] and consequently the Potts model $Z(G, x, y)$ can be determined via Proposition 4.40.

For the transformations in both directions the number of vertices is necessary. For the first direction, this can be calculated from the maximal power of x appearing in the Potts model. For the second direction, the number of vertices equals the cardinality of the polynomial deck. \square

Brešar, Imrich and Klavžar [30] have shown that graph polynomials counting induced subgraphs of an “increasing family” of graphs are reconstructable from the polynomial deck. This includes the result for the clique polynomial and the independence polynomial.

We show that the subgraph counting polynomial of a graph is reconstructable from its corresponding polynomial deck by applying Lemma 4.41.

Theorem 4.42. *Let $G = (V, E)$ be a simple graph with at least three vertices. The subgraph counting polynomial $H(G, v, x, y)$ of G is reconstructable from the polynomial deck $\mathcal{D}_H(G)$.*

Proof. We use the vertex subset expansion of the subgraph counting polynomial (Proposition 4.9):

$$H(G, v, x, y) = \sum_{W \subseteq V} v^{|W|} \cdot Z(G[W], x, y).$$

Analogous to Kelly's Lemma [85], in the sum of the polynomials in the polynomial deck each summand of $H(G, v, x, y)$ including v^i arises $(|V| - i)$ -times, because each subgraph with exactly i vertices is a subgraph of $|V| - i$ vertex-deleted subgraphs.

Hence, only the summands including $v^{|V|}$ are missing, which correspond to the Potts model of G . We can calculate the Potts model of a graph from its subgraph counting polynomial (Proposition 4.9) and consequently the same holds for the polynomial decks, i.e. from $\mathcal{D}_H(G)$ we can calculate $\mathcal{D}_Z(G)$. From this polynomial deck, the Potts model of G can be reconstructed by Lemma 4.41. \square

4.8 Relations to other Graph Polynomials

In this section we give some relations of the edge elimination polynomials to other graph polynomials. While there are only three possible relations, namely one graph polynomial can (proper) generalize the other, both are equivalent, or both can not be related (are "incomparable"), for different graph classes (forests, simple graphs, graphs) different situations may occur.

4.8.1 Relation to the U-polynomial

Noble and Welsh [107, Proposition 5.1] define the U-polynomial $U(G, X, y)$ as an unweighted version of the weighted graph polynomial. Averbouch, Godlin and Makowsky [5, Subsection 1.3] asked, "whether $\xi(G, x, y, z)$ can be obtained as a substitution instance of" the U-polynomial. We show, that for graphs without loops (but with parallel edges allowed) this is the case, otherwise it is not.

Theorem 4.43. *Let $G = (V, E)$ be a graph without loops. The U-polynomial $U(G, X, y)$ generalizes the covered components polynomial $C(G, x, y, z)$ by*

$$C(G, x, y, z) = U(G, X', y + 1) \quad (4.75)$$

where $X' = (x'_1, \dots, x'_{|V|})$ with $x'_1 = x$ and $x'_i = xy^{i-1}z$ for all $i \in \{2, \dots, |V|\}$.

Proof. Substituting the variables in the U-polynomial as given above, where $k_i(G)$ is the number of connected components with exactly i vertices, we get

$$\begin{aligned} U(G, X', y + 1) &= \sum_{A \subseteq E} \prod_{i=1}^{|V|} x'_i{}^{k_i(G\langle A \rangle)} y^{|A| - |V| + k(G\langle A \rangle)} \\ &= \sum_{A \subseteq E} x^{k_1(G\langle A \rangle)} \prod_{i=2}^{|V|} (xy^{i-1}z)^{k_i(G\langle A \rangle)} y^{|A| - |V| + k(G\langle A \rangle)} \\ &= \sum_{A \subseteq E} x^{k(G\langle A \rangle)} y^{|V| - k(G\langle A \rangle)} z^{c(G\langle A \rangle)} y^{|A| - |V| + k(G\langle A \rangle)} \\ &= C(G, x, y, z). \end{aligned} \quad \square$$

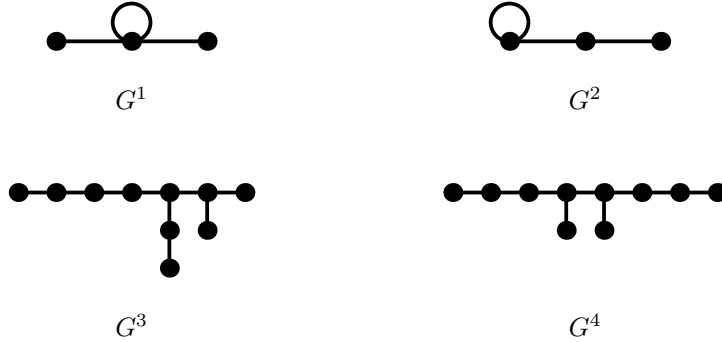


Figure 4.1: G^1 and G^2 are graphs with the same U-polynomial, but different covered components polynomial, G^3 and G^4 are graphs with the same covered components polynomial, but different U-polynomial.

Proposition 4.44. *Let $G = (V, E)$ be a graph and $e \in E$ a loop of G . The U-polynomial $U(G, X, y)$ satisfies*

$$U(G, X, y) = y \cdot U(G_{-e}, X, y). \quad (4.76)$$

From the proposition above it follows that shifting loops does not change the U-polynomial, but it may change the covered components polynomial. An example is the path P_3 on three vertices with a loop at an outer vertex or at the inner vertex. In fact, the minimal example with respect to the number of vertices is a graph consisting of two vertices with two loops at one vertex or one loop at each vertex.

Remark 4.45. The covered components polynomial and the U-polynomial of graphs are not related to each other. This can be observed as follows: The graphs G^1 and G^2 in Figure 4.1, which are paths on 3 vertices with a loop attached to an outer or an inner vertex, have the same U-polynomial, but different covered components polynomial, observe for example the coefficient of $x^2y^2z^2$ that is 0 and 1, respectively. The graphs G^3 and G^4 in Figure 4.1 [139, G^5 and G^6 in Figure 2] have the same covered components polynomial, but different U-polynomial, notice for example the coefficients of x_4x_6 that is 1 and 0, respectively.

4.8.2 Relation to the Chromatic Symmetric Function

The chromatic symmetric function $X(G, X)$, originally defined by Stanley [130], is a specialization of the U-polynomial [107, Theorem 6.1]. For simple graphs it is incomparable with the covered components polynomial.

Remark 4.46. The covered components polynomial and the chromatic symmetric function of simple graphs are not related to each other in general. This can be

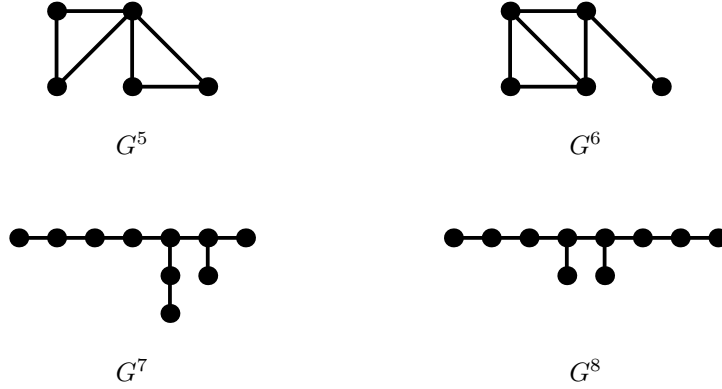


Figure 4.2: G^5 and G^6 are graphs with the same chromatic symmetric function, but different covered components polynomial, G^7 and G^8 are graphs with the same covered components polynomial, but different chromatic symmetric function.

observed as follows: The graphs G^5 and G^6 in Figure 4.2 [130, G and H in Figure 1] have the same chromatic symmetric function, but different covered components polynomial, notice for example the coefficient of x^2y^5z . The graphs G^7 and G^8 in Figure 4.2 [139, G^5 and G^6 in Figure 2] have the same covered components polynomial, but different chromatic symmetric function, because all trees with at most 23 vertices have a unique chromatic symmetric function [101, Section 0, due to Tan].

The situation changes if only trees are considered.

Remark 4.47. The chromatic symmetric function generalizes the covered components polynomial of trees. This can be concluded from the relations of both polynomials to the U-polynomial: For trees, the chromatic symmetric function is equivalent to the U-polynomial (“[107, Theorem 6.1] is easily seen to be reversible for trees” [101, Section 0]), and the U-polynomial generalizes the covered components polynomial (Theorem 4.43).

4.8.3 Relation to the Subgraph Component Polynomial

The subgraph component polynomial $Q(G, x, y)$ is the generating function for the number of vertices and connected components in the induced subgraphs.

It is known that the subgraph component polynomial of the line graph $L(G)$ can be derived from the edge elimination polynomial of G [134, Theorem 23]. We show that for forests the subgraph counting polynomial and the subgraph component polynomial are equivalent to each other.

Theorem 4.48 (Theorem 8 in [138]). *Let $G = (V, E)$ be a forest. The subgraph counting polynomial $H(G, v, x, y)$ and the subgraph component polynomial*

$Q(G, v, x)$ are equivalent graph polynomials related by

$$H(G, v, x, y) = Q(G, v(x+y), \frac{x}{x+y}), \quad (4.77)$$

$$Q(G, v, x) = H(G, v, x, 1-x). \quad (4.78)$$

Proof. We start with the vertex subset expansion of the subgraph counting polynomial in terms of the Potts model (Proposition 4.9). The Potts model of a forest depends only on the number of vertices and the number of connected components (each edge reduces the number of connected components by one), hence we have

$$\begin{aligned} H(G, v, x, y) &= \sum_{W \subseteq V} v^{|W|} \cdot Z(G[W], x, y) \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} (x+y)^{|W|-k(G[W])}. \end{aligned}$$

Then the first equality follows by

$$\begin{aligned} Q(G, v(x+y), \frac{x}{x+y}) &= \sum_{W \subseteq V} (v(x+y))^{|W|} \left(\frac{x}{x+y} \right)^{k(G[W])} \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} (x+y)^{|W|-k(G[W])} \\ &= H(G, v, x, y), \end{aligned}$$

and the second equality follows by

$$\begin{aligned} H(G, v, x, 1-x) &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} (1)^{|V|-k(G[W])} \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} \\ &= Q(G, v, x). \end{aligned} \quad \square$$

Remark 4.49. The subgraph component polynomial and the subgraph counting polynomial of simple graphs are not related to each other. This can be observed as follows: The graphs G^9 and G^{10} in Figure 4.3 [139, G_3 and G_4 in Figure 2] have the same subgraph counting polynomial, but different subgraph component polynomial, notice for example the coefficient of $v^6 x^1$. The graphs G^{11} and G^{12} in Figure 4.3 [7, G_7 and G_8 in Figure 1] have the same subgraph component polynomial, but different chromatic polynomial and hence different subgraph counting polynomial. (For non-simple graphs this follows already from the fact that the subgraph component polynomial does not consider parallel edges and loops, which the subgraph counting polynomial does.)

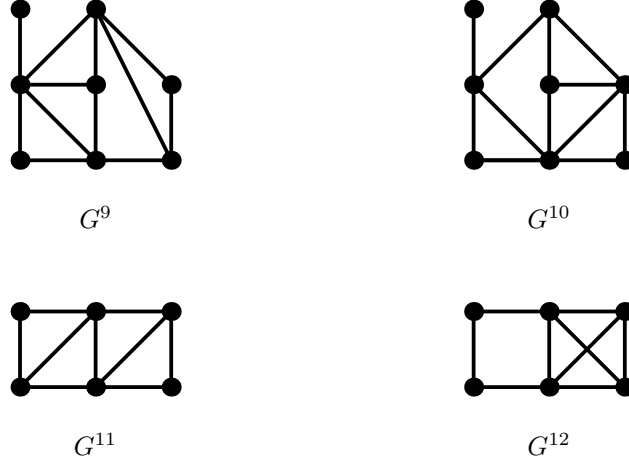


Figure 4.3: G^9 and G^{10} are graphs with the same subgraph counting polynomial, but different subgraph component polynomial, G^{11} and G^{12} are graphs with the same subgraph component polynomial, but different subgraph counting polynomial.

4.8.4 Relation to the Extended Negami Polynomial

The extended Negami polynomial $\tilde{f}(G, t, x, y, z)$ counts vertex mappings with respect to the images of the vertices incident to the edges.

Definition 4.50 (Page 327 of [105]). Let $G = (V, E)$ be a graph. The *extended Negami polynomial* $\tilde{f}(G, t, x, y, z)$ is defined (for $t \in \mathbb{N}$) as

$$\tilde{f}(G, t, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, t\}} \prod_{e \in E} w(e), \quad (4.79)$$

where

$$w(e) = \begin{cases} x + y & \text{if } \forall v \in e: \phi(v) = 1, \\ z + y & \text{if } \exists c \neq 1 \forall v \in e: \phi(v) = c, \\ y & \text{if } \nexists c \forall v \in e: \phi(v) = c. \end{cases} \quad (4.80)$$

The extended Negami polynomial is a proper generalization of the Negami polynomial [104], which is known to be strongly related to the Tutte polynomial [109] and hence also to the Potts model.

For the extended Negami polynomial no more relations than those via its non-extended version are known. We also have no statement connecting the extended Negami polynomial and the trivariate chromatic polynomial, but we can define another specialization of both in two variables, which seems not to be equivalent to the other graph polynomials in two variables we have mentioned, including the Negami polynomial, the Potts model, the bivariate chromatic polynomial and the subgraph component polynomial.

Theorem 4.51. *Let $G = (V, E)$ be a graph. There are specializations of the trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the extended Negami polynomial $\tilde{f}(G, t, x, y, z)$ that equal each other:*

$$\tilde{P}(G, x, x-1, z) = \tilde{f}(G, x, 0, 1, z-1) \quad (4.81)$$

Proof. The trivariate chromatic polynomial can be stated as

$$\tilde{P}(G, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{e \in E} w(e),$$

where

$$w(e) = \begin{cases} z & \text{if } \exists c \leq y \forall v \in e: \phi(v) = c, \\ 1 & \text{else.} \end{cases}$$

Consequently, for $\tilde{P}(G, x, x-1, z)$ the function $w(e)$ changes to

$$w(e) = \begin{cases} z & \text{if } \exists c \leq x-1 \forall v \in e: \phi(v) = c, \\ 1 & \text{else,} \end{cases}$$

which equals the corresponding function for $\tilde{f}(G, x, 0, 1, z-1)$ (with exception that the colors / states are renamed). \square

4.8.5 Relation to the Bivariate Chromatic Polynomial

By Proposition 4.20, the bivariate chromatic polynomial is a specialization of the trivariate chromatic polynomial, and for forests both are equivalent. This is shown by the present author for the covered components polynomial [139, Theorem 38]. Here we show the equivalence of the bivariate and trivariate chromatic polynomial more directly by making use of the fact that for forests the number of edges is determined by the number of vertices and the number of connected components.

Theorem 4.52. *Let $G = (V, E)$ be a forest. The trivariate chromatic polynomial $\tilde{P}(G, x, y, z)$ and the bivariate chromatic polynomial $P(G, x, y, z)$ are equivalent graph polynomials related by*

$$\tilde{P}(G, x, y, z) = (1-z)^{-|V|} \cdot P(G, \frac{x}{1-z}, \frac{y}{1-z}), \quad (4.82)$$

$$P(G, x, y) = \tilde{P}(G, x, y, 0). \quad (4.83)$$

Proof. Using the edge subset expansions of both polynomials (Corollary 4.25), the statement follows by

$$\tilde{P}(G, x, y, z) = \sum_{A \subseteq E} x^{i(G \langle A \rangle)} y^{c(G \langle A \rangle)} (z-1)^{|A|}$$



Figure 4.4: G^{13} and G^{14} are graphs with the same bivariate chromatic polynomial, but different trivariate chromatic polynomial.

$$\begin{aligned}
 &= \sum_{A \subseteq E} x^{i(G \langle A \rangle)} y^{c(G \langle A \rangle)} (1-z)^{|V|-k(G \langle A \rangle)} (-1)^{|A|} \\
 &= (1-z)^{-|V|} \sum_{A \subseteq E} \left(\frac{x}{1-z} \right)^{i(G \langle A \rangle)} \left(\frac{y}{1-z} \right)^{c(G \langle A \rangle)} (-1)^{|A|} \\
 &= (1-z)^{-|V|} \cdot P(G, \frac{x}{1-z}, \frac{y}{1-z}). \quad \square
 \end{aligned}$$

Remark 4.53. The trivariate chromatic polynomial properly generalizes the bivariate chromatic polynomial of simple graphs. This can be concluded from the graphs G^{13} and G^{14} in Figure 4.4 [139, G^1 and G^2 in Figure 1], which have the same bivariate chromatic polynomial, but different covered components polynomials, observe for example the number of vertices of degree 4.

4.8.6 Relation to the Wiener Polynomial

The Wiener polynomial $W(G, x)$ is the generating function for the distance of two vertices.

Definition 4.54 (Equation 1 of [120]). Let $G = (V, E)$ be a connected graph. The *Wiener polynomial* $W(G, q)$ is defined as

$$W(G, q) = \sum_{\{u,v\} \in \binom{V}{2}} q^{d(G,u,v)}, \quad (4.84)$$

where $d(G, u, v)$ is the distance of the vertices u and v in G .

Remark 4.55. The covered components polynomial and the Wiener polynomial of simple graphs are not related to each other. This can be observed as follows: The graphs G^{15} and G^{16} in Figure 4.5, which are a cycle on 4 vertices and a cycle on 3 vertices with an additional pendent edge, have the same Wiener polynomial (2 pairs of vertices of distance 2, all other pairs have distance 1), but different covered components polynomial, observe for example that the coefficient of $x^2 y^3 z^1$ is 0 and 1, respectively. The graphs G^{17} and G^{18} in Figure 4.5 [139, G^5 and G^6 in Figure 2] have the same covered components polynomial, but different Wiener polynomial, notice for example that G^{12} has two vertices of distance 7, which G^{11} does not have.

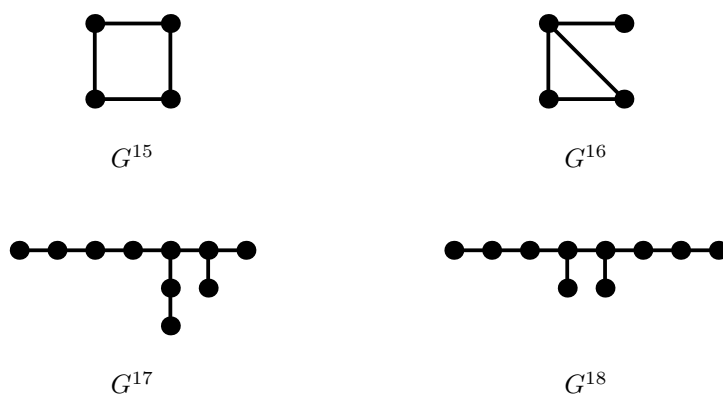


Figure 4.5: G^{15} and G^{16} are graphs with the same Wiener polynomial, but different covered components polynomial, G^{17} and G^{18} are graphs with the same covered components polynomial, but different Wiener polynomial.

Chapter 5

The Generalized Subgraph Counting Polynomial — A Unifying Graph Polynomial

The edge elimination polynomial and the subgraph component polynomial are “most general graph polynomials” with respect to recurrence relations using deletion, contraction and extraction of an edge [4, Theorem 3], and deletion, neighborhood deletion and contraction of a vertex [134, Theorem 22], respectively. That means that every graph polynomial satisfying such recurrence relation can be calculated from them.

In Section 5.1 we define the generalized subgraph counting polynomial, *a unifying graph polynomial* in the sense that it generalizes both the edge elimination polynomial (by generalizing the subgraph counting polynomial) and the subgraph component polynomial. As main result of this chapter we prove a recurrence relation of the newly introduced graph polynomial applicable for hypergraphs.

Some relations to other graph polynomials and some properties are given in Section 5.2 and Section 5.3, respectively. In the last we give some evidence, why the generalized subgraph counting polynomial probably is not “a most general graph polynomial” with respect to the recurrence relation satisfied by itself.

While most of the definitions and statements in this chapter are given for graphs, many of them are also valid in the case of hypergraphs. For this, it may be necessary to generalize known graph polynomials to hypergraphs, their definitions are given accordingly in Appendix A.

5.1 Definition and Recurrence Relation

The generalized subgraph counting polynomial extends the subgraph counting polynomial by additionally counting the number of edges in the subgraphs induced by the vertex set of the considered subgraph.

Definition 5.1. Let $G = (V, E)$ be a hypergraph. The *generalized subgraph counting polynomial* $F(G, v, x, y, z)$ is defined as

$$F(G, v, x, y, z) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|} z^{|E(G[W])|}. \quad (5.1)$$

The generalized subgraph counting polynomial of a hypergraph satisfies a recurrence relation where additional to the already used edge operations (deletion, contraction and extraction of an edge), the deletion of a subset of the vertices incident to an edge is applied.

Theorem 5.2. Let $G = (V, E)$, G^1, G^2 be hypergraphs and $e \in E$ an edge of G . The generalized subgraph counting polynomial $F(G) = F(G, v, x, y, z)$ satisfies

$$\begin{aligned} F(G) &= z \cdot F(G_{-e}) + v^{|e|-1} yz \cdot F(G_{/e}) - v^{|e|-1} yz \cdot F(G_{\ominus e}) \\ &\quad + (z-1) \cdot \sum_{\emptyset \subset B \subseteq e} (-1)^{|B|} \cdot F(G_{\ominus B}), \end{aligned} \quad (5.2)$$

$$F(G^1 \cup G^2) = F(G^1) \cdot F(G^2), \quad (5.3)$$

$$F(K_1) = 1 + vx. \quad (5.4)$$

Proof. The second equality (multiplicativity in components) holds as the subgraphs in different components can be chosen independently from each other, and the third equality (initial value) holds by definition. Therefore, it only remains to show the first equality.

Let $[W', F']$ be the generalized subgraph counting polynomial counting exactly those subgraphs $H = (W, F)$ of G with $W \cap e = W'$ and $F \cap \{e\} = F'$. We have

$$F(G_{-e}) = \sum_{A \subseteq e} [A, \emptyset] + \frac{[e, \emptyset]}{z},$$

$$F(G_{/e}) = [\emptyset, \emptyset] + \frac{[e, \{e\}]}{v^{|e|-1} yz},$$

$$F(G_{\ominus B}) = \sum_{A \subseteq e \setminus B} [A, \emptyset],$$

where $\emptyset \subset B \subseteq e$. The statement follows by

$$\begin{aligned} & z \cdot F(G_{-e}) + v^{|e|-1} yz \cdot F(G_{/e}) - v^{|e|-1} yz \cdot F(G_{\ominus e}) \\ & + (z-1) \cdot \sum_{\emptyset \subset B \subseteq e} (-1)^{|B|} \cdot F(G_{\ominus B}) \\ &= z \cdot \sum_{A \subseteq e} [A, \emptyset] + [e, \emptyset] + v^{|e|-1} yz \cdot [\emptyset, \emptyset] + [e, \{e\}] - v^{|e|-1} yz \cdot [\emptyset, \emptyset] \\ & + (z-1) \cdot \sum_{\emptyset \subset B \subseteq e} (-1)^{|B|} \cdot \sum_{A \subseteq e \setminus B} [A, \emptyset] \end{aligned}$$

$$\begin{aligned}
&= \sum_{A \subseteq e} [A, \emptyset] + [e, \emptyset] + [e, \{e\}] \\
&\quad + (z-1) \cdot \sum_{A \subseteq e} [A, \emptyset] + (z-1) \cdot \sum_{\emptyset \subset B \subseteq e} (-1)^{|B|} \cdot \sum_{A \subseteq e \setminus B} [A, \emptyset] \\
&= F(G),
\end{aligned}$$

where the last identity holds because

$$\begin{aligned}
&\sum_{\emptyset \subset B \subseteq e} (-1)^{|B|} \cdot \sum_{A \subseteq e \setminus B} [A, \emptyset] \\
&= \sum_{\emptyset \subset B \subseteq e} (-1)^{|e| - |\bar{B}|} \cdot \sum_{A \subseteq \bar{B}} [A, \emptyset] \\
&= \sum_{A \subseteq e} [A, \emptyset] (-1)^{|e|} \cdot \left(\sum_{A \subseteq \bar{B} \subseteq e} (-1)^{-|B|} \right) \\
&= \sum_{A \subseteq e} [A, \emptyset] (-1)^{|e|} \cdot \left(\sum_{A \subseteq \bar{B} \subseteq e} (-1)^{-|B|} - (-1)^{|e|} \right) \\
&= \sum_{A \subseteq e} [A, \emptyset] (-1)^{|e|} (-1) (-1)^{|e|} \\
&= - \sum_{A \subseteq e} [A, \emptyset]. \quad \square
\end{aligned}$$

Corollary 5.3. *Let $G = (V, E)$ be graphs and $e = \{u, v\}, f = \{v\} \in E$ a link and a loop of G . The generalized subgraph counting polynomial $F(G) = F(G, v, x, y, z)$ satisfies*

$$\begin{aligned}
F(G) &= z \cdot F(G_{-e}) + vyz \cdot F(G_{/e}) + (z-1-vyz) \cdot F(G_{\dagger e}) \\
&\quad + (1-z) \cdot F(G_{\ominus u}) + (1-z) \cdot F(G_{\ominus v}), \tag{5.5}
\end{aligned}$$

$$= (z+yz) \cdot F(G_{-f}) + (1-z-yz) \cdot F(G_{\dagger f}). \tag{5.6}$$

Proof. The statement follows directly from the theorem above. \square

5.2 Relations

The generalized subgraph counting polynomial is a generalization of both the subgraph counting polynomial and the subgraph component polynomial. While the first fact follows directly from the definition, for the second one we have to show that we can extract the terms corresponding to subgraphs including all edges appearing in the induced subgraph on the same set of vertices.

Proposition 5.4. *Let $G = (V, E)$ be a hypergraph. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ generalizes the subgraph counting polynomial $H(G, v, x, y)$ by*

$$H(G, v, x, y) = F(G, v, x, y, 1). \tag{5.7}$$

Theorem 5.5. *Let $G = (V, E)$ be a graph. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ generalizes the subgraph component polynomial $Q(G, v, x)$ by*

$$Q(G, v, x) = F\left(G, v, x, \frac{1}{z}, z\right)\Big|_{z=0}. \quad (5.8)$$

Proof. Substituting y by $\frac{1}{z}$ in the definition of the generalized subgraph counting polynomial, we get

$$\begin{aligned} F\left(G, v, x, \frac{1}{z}, z\right) &= \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} \left(\frac{1}{z}\right)^{|F|} z^{|E(G[W])|} \\ &= \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} z^{|E(G[W])| - |F|}. \end{aligned}$$

Substituting z by 0, all summands corresponding to subgraphs with $|E(G[W])| \neq |F|$ equal 0, hence for each vertex subset W only one subgraph is counted (in the case $|E(G[W])| = |F|$) and therefore the statement follows:

$$\begin{aligned} F\left(G, v, x, \frac{1}{z}, z\right)\Big|_{z=0} &= \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} [|E(G[W])| = |F|] \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} \\ &= Q(G, v, x). \quad \square \end{aligned}$$

Corollary 5.6. *Let $G = (V, E)$ be a graph and $e = \{u, v\}, f = \{v\} \in E$ a link and a loop of G . The subgraph component polynomial $Q(G) = Q(G, v, x)$ satisfies*

$$Q(G) = v \cdot Q(G_{/e}) - (1 + v) \cdot Q(G_{\dagger e}) + Q(G_{\ominus u}) + Q(G_{\ominus v}), \quad (5.9)$$

$$= Q(G_{-f}). \quad (5.10)$$

Proof. The statement follows directly by applying the relation to the generalized subgraph counting polynomial (Theorem 5.5) in its recurrence relation (Corollary 5.3). \square

Remark 5.7. The generalized subgraph counting polynomial is a proper generalization of both the subgraph counting polynomial and the subgraph component polynomial. This results from the pairs of non-isomorphic graphs having one of the graph polynomials in common, but not the other one (Remark 4.49).

In contrast, for forests all three graph polynomials are equivalent.

Theorem 5.8. *Let $G = (V, E)$ be a forest. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ and the subgraph component polynomial $Q(G, v, x)$ are equivalent graph polynomials related by*

$$Q(G, v, x) = F\left(G, v, x, \frac{1}{z}, z\right)\Big|_{z=0}, \quad (5.11)$$

$$F(G, v, x, y, z) = Q(G, v(x+y)z, \frac{x}{(x+y)z}). \quad (5.12)$$

Proof. The first equality results from the situation for graphs (Theorem 5.5). For the second equality we can argue, that from the term of the subgraph component polynomial (counting $|W|$ and $k(G[W])$), the number of edges in the induced subgraph can be determined ($|E(G[W])| = |W| - k(G[W])$). Then, by assuming that an edge may be in a subgraph or not, we can extend the summation to all subgraphs, where the number of connected components in the subgraph increases by one for each missing edge. That means, starting with the definition of the subgraph component polynomial, the statement follows by

$$\begin{aligned} & Q(G, v(x+y)z, \frac{x}{(x+y)z}) \\ &= \sum_{W \subseteq V} (v(x+y)z)^{|W|} \left(\frac{x}{(x+y)z} \right)^{k(G[W])} \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} (x+y)^{|W| - k(G[W])} z^{|W| - k(G[W])} \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} (x+y)^{|E(G[W])|} z^{|E(G[W])|} \\ &= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} \left(\sum_{F \subseteq E(G[W])} x^{|E(G[W] \setminus F)|} y^{|F|} \right) z^{|E(G[W])|} \\ &= \sum_{W \subseteq V} \sum_{F \subseteq E(G[W])} v^{|W|} x^{k(G[W]) + |E(G[W] \setminus F)|} y^{|F|} z^{|E(G[W])|} \\ &= \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|} z^{|E(G[W])|} \\ &= F(G, v, x, y, z). \quad \square \end{aligned}$$

The equivalence for forests of the subgraph counting polynomial and its generalized version can be concluded from the theorem above and its equivalence to the subgraph component polynomial in the case of forests (Theorem 4.48). However, we state it directly because of the maybe interesting proof.

Theorem 5.9. *Let $G = (V, E)$ be a forest. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ and the subgraph counting polynomial $H(G, v, x, y)$ are equivalent graph polynomial related by*

$$H(G, v, x, y) = F(G, v, x, y, 1), \quad (5.13)$$

$$F(G, v, x, y, z) = H(G, v, x, xz + yz - x). \quad (5.14)$$

Proof. The first equality follows directly from the definitions. For the second equality we compare the recurrence relation for both graph polynomials in case of pendent edges, which are edges such that at least one incident vertex has degree 1.

Because each non-empty forest has at least one pendent edge, a graph polynomial of a forest can be calculated recursively by a recurrence relation for pendent edges (together with the multiplicativity with respect to components and an initial value).

Let $e = \{u, v\} \in E$ be a pendent edge of G with $\deg(G, u) = 1$. From the recurrence relation for the generalized subgraph counting polynomial $F(G) = F(G, v, x, y, z)$ of an arbitrary edge (Theorem 5.2) we get

$$\begin{aligned} F(G) &= z \cdot F(G_{-e}) + vyz \cdot F(G_{/e}) + (z - 1 - vyz) \cdot F(G_{\dagger e}) \\ &\quad + (1 - z) \cdot F(G_{\ominus u}) + (1 - z) \cdot F(G_{\ominus v}) \\ &= z(1 + vx) \cdot F(G_{\ominus u}) + vyz \cdot F(G_{\ominus u}) + (z - 1 - vyz) \cdot F(G_{\ominus u \ominus v}) \\ &\quad + (1 - z) \cdot F(G_{\ominus u}) + (1 - z)(1 + vx) \cdot F(G_{\ominus v}) \\ &= [vxz + vyz + 1] \cdot F(G_{\ominus u}) + [vx - vxz - vyz] \cdot F(G_{\ominus u \ominus v}), \end{aligned}$$

and for $H(G) = H(G, v, x, y) = F(G, v, x, y, 1)$ it follows

$$H(G) = [vx + vy + 1] \cdot H(G_{\ominus u}) + [-vy] \cdot H(G_{\ominus u \ominus v}).$$

Therefore, $H(G, v, x, xz + yz - x)$ and $F(G, v, x, y, z)$ satisfy the same recurrence relation and consequently the second equality holds. \square

In the literature there is another graph polynomial generalizing both the edge elimination polynomial and the subgraph component polynomial, the homomorphism polynomial $H(G, k, x, y, z)$ [59, page 1044] defined by Garijo, Goodall and Nešetřil. It is defined as the number of homomorphisms from G to the graph $H_{k,x,y,z}$, $H(G, k, x, y, z) = \text{hom}(G, H_{k,x,y,z})$, where $H_{k,x,y,z}$ is the join of a K_k^z with the disjoint union of y copies of K_x^z , where K_n^l is a complete graph on n vertices with l loops attached at each vertex. The authors give a vertex subset expansion of the graph polynomial in terms of the Potts models of induced graphs, which can be rewritten such that the difference to the generalized subgraph counting polynomial becomes more clear.

Proposition 5.10 (Page 1044 in [59]). *Let $G = (V, E)$ be a graph. The homomorphism polynomial $h(G, k, x, y, z)$ satisfies*

$$h(G, k, x, y, z) = \sum_{W \subseteq V} k^{|V|-|W|} y^{k(G[W])} P(G[W], x, z) \quad (5.15)$$

$$= \sum_{H=(W,F) \subseteq G} k^{|V|-|W|} y^{k(G[W])} x^{k(H)} z^{|F|}. \quad (5.16)$$

Therefore, in addition to renaming variables, the difference between the generalized subgraph counting polynomial and the homomorphism polynomial is that the first counts the edges and the second counts the connected components in the subgraph induced by the vertex set of the considered subgraph.

But this fact gives no straightforward insight into the relation of both graph polynomials and hence this is an open problem (Question 6).

5.3 Properties

5.3.1 Polynomial Reconstructability

The generalized subgraph counting polynomial of a graph is reconstructable from its polynomial deck by the same line of arguments valid for the non-generalized version (Theorem 4.42), caused by a similar vertex subset expansion as a sum over Potts models.

Proposition 5.11. *Let $G = (V, E)$ be a graph. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ generalizes the Potts model $Z(G, x, y)$ by*

$$Z(G, x, y) = [v^{|V|}](F(G, v, x, y, 1)), \quad (5.17)$$

and has the vertex subset expansion

$$F(G, v, x, y, z) = \sum_{W \subseteq V} v^{|W|} z^{|E(G[W])|} \cdot Z(G[W], x, y). \quad (5.18)$$

Theorem 5.12. *Let $G = (V, E)$ be a simple graph with at least three vertices. The generalized subgraph counting polynomial $F(G, v, x, y, z)$ of G is reconstructable from the polynomial deck $\mathcal{D}_F(G)$.*

Proof. We use the vertex subset expansion of the generalized subgraph counting polynomial (Proposition 5.11):

$$F(G, v, x, y, z) = \sum_{W \subseteq V} v^{|W|} z^{|E(G[W])|} \cdot Z(G[W], x, y).$$

Analogous to Kelly's Lemma [85], in the sum of the polynomials in the polynomial deck each summand of $F(G, v, x, y, z)$ including v^i arises $(|V| - i)$ -times, because each subgraph with exactly i vertices is a subgraph of $|V| - i$ vertex-deleted subgraphs.

Hence, only the summands including $v^{|V|}$ are missing, which correspond to the Potts model of G (the factor $z^{|E|}$ can be determined as $\deg_y(Z(G, x, y))$). We can calculate the Potts model of a graph from its extended subgraph counting polynomial (Proposition 5.11) and consequently the same holds for the polynomial decks, i.e. from $\mathcal{D}_F(G)$ we can calculate $\mathcal{D}_Z(G)$. From this polynomial deck, the Potts model of G can be reconstructed by Lemma 4.41. \square

5.3.2 Non-isomorphic Graphs with Coinciding Generalized Subgraph Counting Polynomial

We have already shown that the generalized subgraph counting polynomial is a proper generalization of the subgraph counting polynomial and the subgraph component polynomial. This can also be seen from the fact that the last two do not distinguish all simple graphs with 8 vertices, which is done by the first.

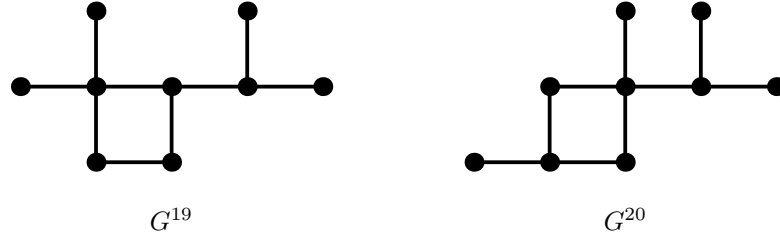


Figure 5.1: G^{19} and G^{20} are two non-isomorphic graphs with the same extended subgraph counting polynomial.

Remark 5.13. The generalized subgraph counting polynomial distinguishes simple graphs with less than 9 vertices. The graphs G^{19} and G^{20} of Figure 5.1 are the minimal non-isomorphic graphs (minimal with respect to the number of vertices) with the same generalized subgraph counting polynomial.

For forests, all three mentioned graph polynomials are equivalent (Theorem 5.8 and Theorem 5.9). Therefore, the pairs of non-isomorphic trees with the same covered components polynomial have also the same generalized subgraph counting polynomial. For all those pairs with up to 12 vertices, see [139, Figure 2 and Figure 3].

5.3.3 Not Necessarily a “Most General” Graph Polynomial

The edge elimination polynomial and the subgraph counting polynomial are the most general graph polynomials with respect to those graph polynomials satisfying a similar recurrence relation. However, it seems unlikely that the same holds for the generalized subgraph counting polynomial.

Definition 5.14. Let $G = (V, E)$, G^1, G^2 be a graphs and $e = \{u, v\}, f = \{w\} \in E$ a link and a loop of G . The (non-invariant) graph polynomial $\Xi(G) = \Xi(G, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, x)$ is defined by

$$\Xi(G) = \alpha \cdot \Xi(G_{-e}) + \beta \cdot \Xi(G_{/e}) + \gamma \cdot \Xi(G_{\dagger e}) \quad (5.19)$$

$$+ \delta \cdot \Xi(G_{\ominus u}) + \delta \cdot \Xi(G_{\ominus v}), \quad (5.20)$$

$$\Xi(G) = \epsilon \cdot \Xi(G_{-f}) + \zeta \cdot \Xi(G_{\dagger f}), \quad (5.21)$$

$$\Xi(E_n) = x^n. \quad (5.22)$$

We determine sufficient conditions of the parameters α, \dots, ζ , such that the graph polynomial $\Xi(G)$ is an invariant.

Remark 5.15. We have already assumed that the coefficients of $\Xi(G_{\ominus u})$ and $\Xi(G_{\ominus v})$ are equal (here δ), because this is obviously a necessary condition for an invariant graph polynomial.

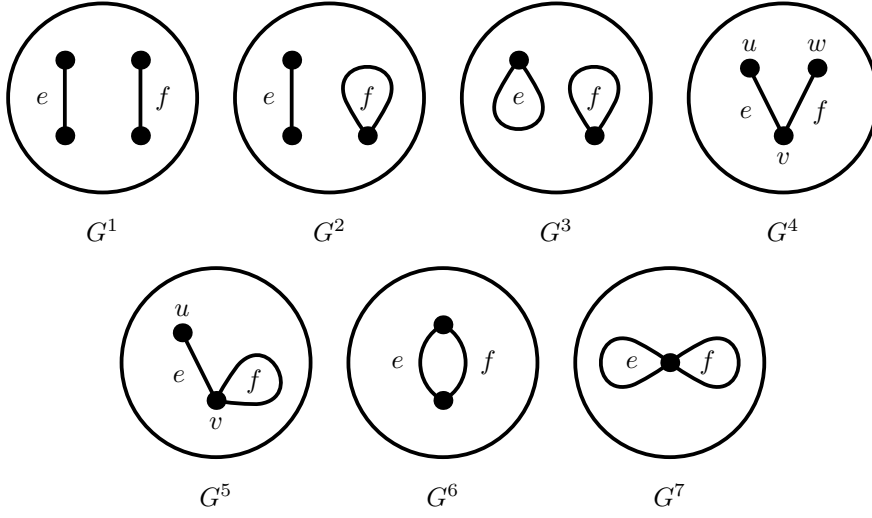


Figure 5.2: Different cases for the elimination of two edges e and f .

By Theorem 3.16 it follows that the graph polynomial $\Xi(G)$ is multiplicative in components.

We have to determine under which conditions the application of the recurrence relation for two edges is interchangeable. This approach and also the following analysis are analogous to those given by Averbouch, Godlin and Makowsky [3, Section 3.3; 4, Section 2].

Theorem 5.16. *The graph polynomial $\Xi(G, \alpha, \beta, \gamma, \delta, \epsilon, \zeta)$ is an invariant graph polynomial if the following equations are satisfied:*

$$(\alpha - 1)\gamma = (\beta + \delta)\delta, \quad (5.23)$$

$$(\alpha - 1)\zeta = \delta(\epsilon - 1), \quad (5.24)$$

$$(\beta + \delta)\zeta = \gamma(\epsilon - 1). \quad (5.25)$$

Proof. We have to check all situations of the mutual location of two edges, either these are two links, a link and a loop, or two loops. The cases are displayed in Figure 5.2.

Let EF^i be $\Xi(G^i)$ calculated by eliminating first the edge e and second the edge f and let FE^i be $\Xi(G^i)$ calculated by eliminating first the edge f and second the edge e . The conditions must be determined, under which the arising graph polynomial is independent of the order of the elimination of the edges e and f , i. e. $EF^i = FE^i = 0$ for $i = 1, \dots, 7$.

If the two edges have no common vertex, as in the graphs G^1 , G^2 and G^3 , or are symmetrically, as in the graphs G^6 and G^7 , the graphs occurring and their prefactors do not depend on the order of the operated edge. Hence, the resulting polynomials EF^i and FE^i are identical for $i = 1, 2, 3, 6, 7$.

It remains to check the cases for $i = 4, 5$, where we write G_s instead of $\Xi(G_s)$ for the sake of brevity.

The graph G^4 contains the edges $e = \{u, v\}$ and $f = \{v, w\}$. We have

$$\begin{aligned}
 EF^4 &= \alpha \cdot G_{-e} + \beta \cdot G_{/e} + \gamma \cdot G_{\dagger e} + \delta \cdot G_{\ominus u} + \delta \cdot G_{\ominus v} \\
 &= \alpha \cdot [\alpha \cdot G_{-e-f} + \beta \cdot G_{-e/f} + \gamma \cdot G_{-e\dagger f} + \delta \cdot G_{-e\ominus v} + \delta \cdot G_{-e\ominus w}] \\
 &\quad + \beta \cdot [\alpha \cdot G_{/e-f} + \beta \cdot G_{/e/f} + \gamma \cdot G_{/e\dagger f} + \delta \cdot G_{/e\ominus v} + \delta \cdot G_{/e\ominus w}] \\
 &\quad + \gamma \cdot G_{\dagger e} \\
 &\quad + \delta \cdot [\alpha \cdot G_{\ominus u-f} + \beta \cdot G_{\ominus u/f} + \gamma \cdot G_{\ominus u\dagger f} + \delta \cdot G_{\ominus u\ominus v} + \delta \cdot G_{\ominus u\ominus w}] \\
 &\quad + \delta \cdot G_{\ominus v}
 \end{aligned}$$

and

$$\begin{aligned}
 FE^4 &= \alpha \cdot G_{-f} + \beta \cdot G_{/f} + \gamma \cdot G_{\dagger f} + \delta \cdot G_{\ominus v} + \delta \cdot G_{\ominus w} \\
 &= \alpha \cdot [\alpha \cdot G_{-f-e} + \beta \cdot G_{-f/e} + \gamma \cdot G_{-f\dagger e} + \delta \cdot G_{-f\ominus u} + \delta \cdot G_{-f\ominus v}] \\
 &\quad + \beta \cdot [\alpha \cdot G_{/f-e} + \beta \cdot G_{/f/e} + \gamma \cdot G_{/f\dagger e} + \delta \cdot G_{/f\ominus u} + \delta \cdot G_{/f\ominus v}] \\
 &\quad + \gamma \cdot G_{\dagger f} \\
 &\quad + \delta \cdot G_{\ominus v} \\
 &\quad + \delta \cdot [\alpha \cdot G_{\ominus w-e} + \beta \cdot G_{\ominus w/e} + \gamma \cdot G_{\ominus w\dagger e} + \delta \cdot G_{\ominus w\ominus u} + \delta \cdot G_{\ominus w\ominus v}].
 \end{aligned}$$

Using the commutativity of the following operations

$$\begin{aligned}
 G_{-e-f} &= G_{-f-e}, & G_{-e/f} &= G_{/f-e}, & G_{-e\ominus w} &= G_{\ominus w-e}, \\
 G_{/e-f} &= G_{-f/e}, & G_{/e/f} &= G_{/f/e}, & G_{/e\ominus w} &= G_{\ominus w/e}, \\
 G_{\ominus u-f} &= G_{-f\ominus u}, & G_{\ominus u/f} &= G_{/f\ominus u}, & G_{\ominus u\ominus w} &= G_{\ominus w\ominus u},
 \end{aligned}$$

and the following identities

$$\begin{aligned}
 G_{\ominus v} &= G_{-e\ominus v} = G_{-f\ominus v}, \\
 G_{\ominus u\ominus v} &= G_{/e\ominus v} = G_{\dagger e} = G_{-f\dagger e}, \\
 G_{\ominus v\ominus w} &= G_{-e\dagger f} = G_{\dagger f} = G_{\ominus v\ominus w}, \\
 G_{\ominus u\ominus v\ominus w} &= G_{/e\dagger f} = G_{/f\dagger e} = G_{\ominus u\dagger f} = G_{\ominus w\dagger e},
 \end{aligned}$$

we get as a necessary condition for a graph invariant that

$$EF_4 - FE_4 = (G_{\ominus u\ominus v} - G_{\ominus v\ominus w}) \cdot (-\alpha\gamma + \beta\delta + \gamma + \delta^2) = 0,$$

which equals

$$(\alpha - 1)\gamma = (\beta + \delta)\delta. \quad (5.26)$$

The graph G^5 contains the edges $e = \{u, v\}$ and $f = \{v\}$. We have

$$EF^5 = \alpha \cdot G_{-e} + \beta \cdot G_{/e} + \gamma \cdot G_{\dagger e} + \delta \cdot G_{\ominus u} + \delta \cdot G_{\ominus v}$$

$$\begin{aligned}
&= \alpha \cdot [\epsilon \cdot G_{-e-f} + \zeta \cdot G_{-e\ddagger f}] \\
&\quad + \beta \cdot [\epsilon \cdot G_{/e-f} + \zeta \cdot G_{/e\ddagger f}] \\
&\quad + \gamma \cdot G_{\ddagger e} \\
&\quad + \delta \cdot [\epsilon \cdot G_{\ominus u-f} + \zeta \cdot G_{\ominus u\ddagger f}] \\
&\quad + \delta \cdot G_{\ominus v}
\end{aligned}$$

and

$$\begin{aligned}
FE^5 &= \epsilon \cdot G_{-f} + \zeta \cdot G_{\ddagger f} \\
&= \epsilon \cdot [\alpha \cdot G_{-f-e} + \beta \cdot G_{-f/e} + \gamma \cdot G_{-f\ddagger e} + \delta \cdot G_{-f\ominus u} + \delta \cdot G_{-f\ominus v}] \\
&\quad + \zeta \cdot G_{\ddagger f}.
\end{aligned}$$

Using the commutativity of the following operations

$$\begin{aligned}
G_{-e-f} &= G_{-f-e}, & G_{-e/f} &= G_{/f-e}, & G_{/e-f} &= G_{-f/e}, \\
G_{/e/f} &= G_{/f/e}, & G_{\ominus u-f} &= G_{-f\ominus u}, & G_{\ominus u/f} &= G_{/f\ominus u},
\end{aligned}$$

and the following identities

$$\begin{aligned}
G_{\ominus v} &= G_{-e\ddagger f} = G_{\ddagger f} = G_{-e\ominus v} = G_{-f\ominus v}, \\
G_{\ominus u\ominus v} &= G_{/e\ominus v} = G_{\ddagger e} = G_{-f\ddagger e} = G_{\ominus u\ddagger f} = G_{/e\ddagger f} = G_{/f\ddagger e},
\end{aligned}$$

we get as a necessary condition for a graph invariant that

$$\begin{aligned}
EF_5 - FE_5 &= G_{\ominus u\ominus v} \cdot (\beta\zeta + \gamma + \delta\zeta - \epsilon\gamma) \\
&\quad + G_{\ominus v} \cdot (\alpha\zeta + \delta - \delta\epsilon - \zeta) = 0,
\end{aligned}$$

which equals

$$\begin{aligned}
(\alpha - 1)\zeta &= \delta(\epsilon - 1), \\
(\beta + \delta)\zeta &= \gamma(\epsilon - 1).
\end{aligned}$$

□

The edge elimination polynomial corresponds to the case $\delta = 0$, $\epsilon = \alpha + \beta$ and $\zeta = \gamma$. Thereby, the second equation is fulfilled and the third one equals the first one. The first equation has in fact two different solutions, $\alpha = 1$ and $\gamma = 0$. Hence, to show that this graph polynomial is “a most general” one satisfying such kind of recurrence relation, it has been necessary to argument that the case $\alpha = 1$ is more general than $\gamma = 0$. This holds, because the second case is equivalent to the Potts model [4, Equation 21].

While the generalized subgraph counting polynomial also satisfies the equations given in the theorem above, there are much more solutions for them. Whether each of these solutions is a specialization of or equivalent to the generalized subgraph counting polynomial is an open problem (Question 7), but at least it seems to be unlikely. Therefore, to attack this problem, it seems appropriate to look for solutions corresponding to graph polynomials distinguishing non-isomorphic graphs with the same generalized subgraph counting polynomial, for example G^1 and G^2 in Figure 5.1.

Chapter 6

Conclusion

The aim of this dissertation is to make some progress to a better understanding of graph polynomials and their relations. This is done by

- collecting several concepts used to define graph polynomials and stating relations between them (Chapter 3),
- introducing graph polynomials equivalent to the edge elimination polynomial and thereby linking a graph polynomial defined by a recurrence relation to the counting of (spanning) subgraphs and colorings (Chapter 4),
- defining a graph polynomial that unifies several of the well-known graph polynomials (Chapter 5).

We think some evidence is given that the investigation of equivalent graph polynomials can be very useful, although or rather because these are mathematically the same.

We have restricted our investigation to graph polynomials associated to hypergraphs with a constant number of variables. Therefore, neither graph polynomials associated to other graph-like structures, for example directed graphs, graph embeddings and knots, nor multivariate graph polynomials, where usually a variable for each vertex or edge is used, are considered here. Furthermore, the topic of this thesis has been concentrated on graph polynomials that can be defined by or satisfy a recurrence relation.

For sure, this sole work can not cover all topics related to graph polynomials, not even with the mentioned restrictions. But it seems that graph polynomials get a lot of attention in the last years, especially from PhD students. Hence, together with the theses of Averbouch [3], Csikvari [43] and Hoffmann [78], whose topics are all mostly mutually disjoint, there has been some progress “towards a general theory of graph polynomials” [100].

In the following Section 6.1 we present graphically our current knowledge about the relations between some graph polynomials. Some open problems mentioning unknown relations and other questions for further research are given in Section 6.2.

6.1 An Overview about Graph Polynomials

We summarize the most of our knowledge about the relations between different graph polynomials in Figure 6.1. There we display a “graph of graph polynomials”, where the vertices represent graph polynomials and where an directed edge from graph polynomial A to graph polynomial B means that B can be calculated from A in the case of graphs.

Please note the following remarks for the dashed edges:

- the relation from the U-polynomial to the covered components polynomial holds only for simple graphs,
- for the calculation of the Potts model and the Negami polynomial from the Tutte polynomial, the number of vertices is necessary.

6.2 Open Problems

We list some of the questions still open, mostly regarding relations of the mentioned graph polynomials. Some of them arise from some “missing edges” in Figure 6.1 and could be answered in the negative by finding some specific pair of non-isomorphic graphs having some graph polynomial in common, but not some other one.

All recurrence relations discussed in this work are linear, with exception of one given for the characteristic polynomial in Equation (3.57).

Question 1. *Are there other non-linear recurrence relations for graph polynomials?*

The chromatic symmetric function can be generalized to a “bad coloring” version, which we denote as bad coloring symmetric function $\tilde{X}(G, X, z)$. This is equivalent to the U-polynomial [107, Theorem 6.2].

Question 2. *Is there a “direct combinatorial” relation between the bad coloring symmetric function and the trivariate chromatic polynomial?*

There are some graph polynomials generalized by both the trivariate chromatic polynomial and extended Negami polynomial. But nothing is known about their relation to each other.

Question 3. *Are the trivariate chromatic polynomial and the extended Negami polynomial related (for graphs / forests)?*

We know that there are pairs of non-isomorphic trees with the same covered components polynomial but different Wiener polynomial, but are not aware of examples for the other way around.

Question 4. *Is the Wiener polynomial a generalization of the covered components polynomial for forests?*

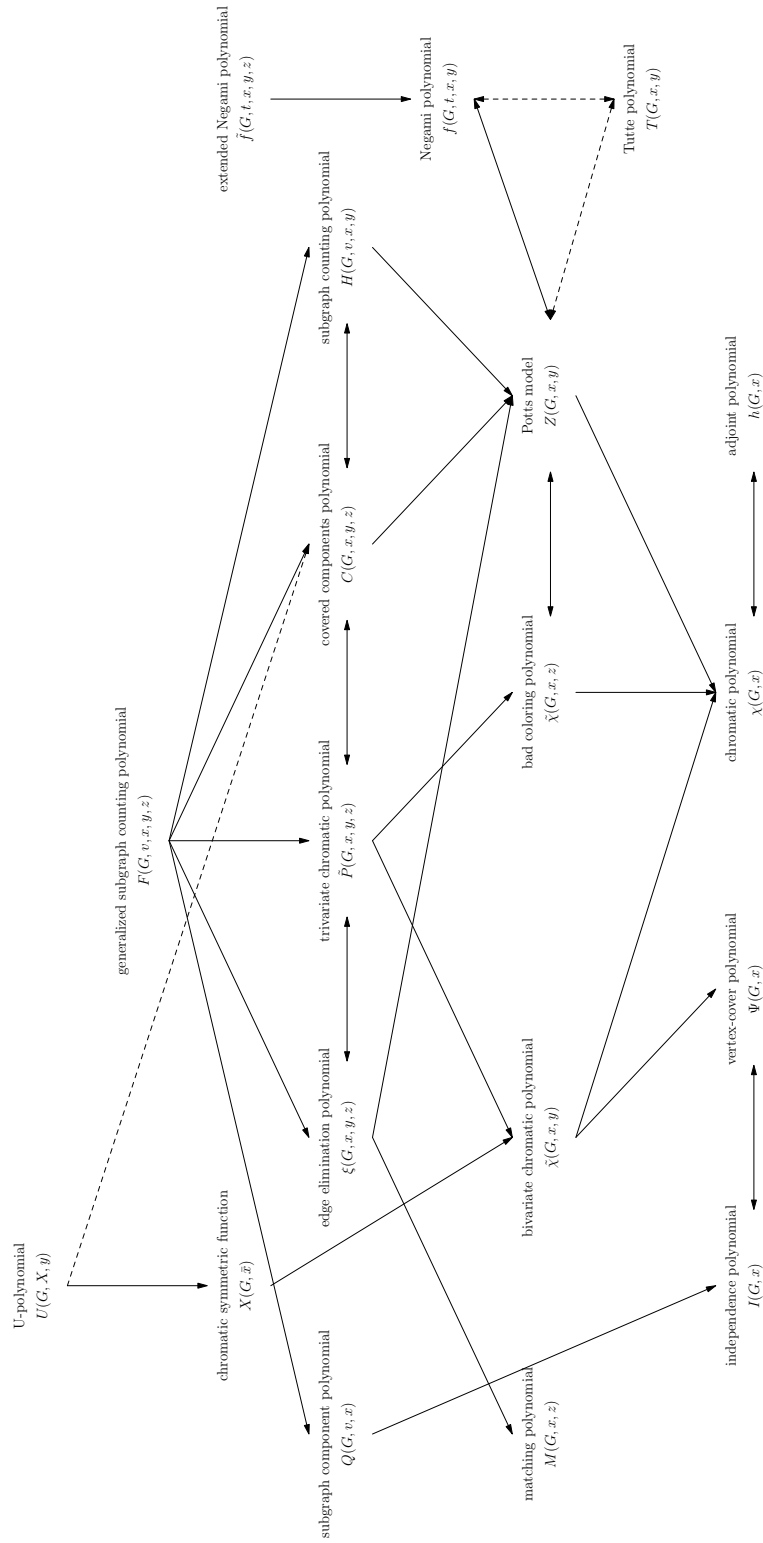


Figure 6.1: A “graph of graph polynomials” presenting the relations between different graph polynomials.

The generalized subgraph counting polynomial generalizes the trivariate chromatic polynomial. Can the generalized subgraph counting polynomial be defined in terms of colorings which makes this relation obvious?

Question 5. *Is there a coloring expansion of the generalized subgraph counting polynomial?*

There are some other graph polynomials generalizing the same graph polynomials as the generalized subgraph counting polynomial. It seems that the relation between these graph polynomials is not yet considered.

Question 6. *Are the generalized subgraph counting polynomial, the extended Negami polynomial, and the homomorphism polynomial related to each other (for graphs / forests)?*

Question 7. *Is the generalized subgraph counting polynomial a “most general graph polynomial” with respect to the recurrence relation it satisfies?*

If we restrict the recurrence relations to those with $\delta = 0$, is then the edge elimination polynomial also most general or are by the independence of the factors for a link and a loop other graph polynomials possible?

Question 8. *Is there a most general graph polynomial satisfying Theorem 5.16 in the case $\delta = 0$?*

Appendix A

List of Graph Polynomials

In this appendix we list some literature and a definition for the used graph polynomials. If possible, the definitions are given in such a way, that they can easily be generalized to hypergraphs.

Adjoint polynomial $h(G, x)$

Literature: [52; 97; 159; 160; 162].

Definition A.1 (Section 11.1 in [52]). Let $G = (V, E)$ be a simple graph. The *adjoint polynomial* $h(G, x)$ is defined as

$$h(G, x) = \sum_{\pi \in \Pi_I(\bar{G})} x^{|\pi|}, \quad (\text{A.1})$$

where $\bar{G} = (V, \binom{V}{2} \setminus E)$.

Bad Coloring Polynomial $\tilde{\chi}(G, x, y)$

Literature: [33; 54; 66; 129; 140; 142; 147; 149].

Definition A.2 (Page 63 in [147]). Let $G = (V, E)$ be a graph. The *bad coloring polynomial* $\tilde{\chi}(G, x, z)$ is defined (for $x \in \mathbb{N}$) as

$$\tilde{\chi}(G, x, z) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \forall v \in e: \phi(v)=c}} z. \quad (\text{A.2})$$

Bad Coloring Symmetric Function $\tilde{X}(G, X, z)$

Literature: [38; 107; 122; 131].

Definition A.3 (Definition 3.1 in [131]). Let $G = (V, E)$ be a graph. The *bad coloring symmetric function* $\tilde{X}(G, X, z)$ is defined as

$$\tilde{X}(G, X, z) = \sum_{\text{coloring } \phi} \prod_{v \in V} x_{\phi(v)} \prod_{\substack{e \in E \\ \exists c \forall v \in e: \phi(v)=c}} 1 + z. \quad (\text{A.3})$$

Bivariate Chromatic Polynomial $P(G, x, y)$

Literature: [4; 5; 49; 50; 73; 77; 78].

Definition A.4 (Section 1 in [50]). Let $G = (V, E)$ be a graph. The *bivariate chromatic polynomial* $P(G, x, y)$ is defined as

$$P(G, x, y) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \leq y \forall v \in e: \phi(v)=c}} 0. \quad (\text{A.4})$$

Characteristic Polynomial $\phi(G, x)$

Literatur[44; 67; 80; 117; 119; 136]

Definition A.5 (Section 0.1 in [44]). Let $G = (V, E)$ be a graph with adjacency matrix $A(G)$. The *characteristic polynomial* $\phi(G, x)$ is defined as

$$\phi(G, x) = \det(xI - A(G)), \quad (\text{A.5})$$

where I is the identity matrix of format $|V| \times |V|$.

Chromatic Polynomial $\chi(G, x)$

Literature: [8; 13; 14; 16; 17; 18; 19; 20; 23; 37; 52; 103; 112; 114; 115; 128; 141; 151; 152; 153; 161].

Definition A.6 ([18], Section 2 in [151]). Let $G = (V, E)$ be a graph. The *chromatic polynomial* $\chi(G, x)$ is defined (for $x \in \mathbb{N}$) as

$$\chi(G, x) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \forall v \in e: \phi(v)=c}} 0. \quad (\text{A.6})$$

Chromatic Symmetric Function $X(G, X)$

Literature: [38; 101; 130; 131].

Definition A.7 (Definition 2.1 in [130]). Let $G = (V, E)$ be a graph. The *chromatic symmetric function* $X(G, X)$ is defined as

$$X(G, X) = \sum_{\text{proper coloring } \phi} \prod_{v \in V} x_{\phi(v)}. \quad (\text{A.7})$$

Clique Polynomial $C(G, x)$

Literature: [58; 69; 75; 96].

Definition A.8 (Definition 2.1 in [75]). Let $G = (V, E)$ be a simple graph. The *clique polynomial* is defined as

$$C(G, x) = \sum_{i=0}^{|V|} c_i(G) x^i, \quad (\text{A.8})$$

where $c_i(G)$ is the number of complete subgraphs of G with exactly i vertices.

Covered Components Polynomial $C(G, x, y, z)$

Literature: [3; 6; 139].

Definition A.9 (Definition 3 in [139]). Let $G = (V, E)$ be a hypergraph. The *covered components polynomial* $C(G, x, y, z)$ is defined as

$$C(G, x, y, z) = \sum_{A \subseteq E} x^{k(G\langle A \rangle)} y^{|A|} z^{c(G\langle A \rangle)}. \quad (\text{A.9})$$

Edge Coloring Polynomial $\chi'(G, x)$

Literature: [70].

Definition A.10 (Section 2 in [70]). Let $G = (V, E)$ be a graph. The *edge coloring polynomial* $\chi'(G, x)$ is defined as

$$\chi'(G, x) = \sum_{\phi: E \rightarrow \{1, \dots, x\}} \prod_{v \in V} \gamma(v), \quad (\text{A.10})$$

with

$$\gamma(v) = \begin{cases} 0 & \text{if } \exists e_1, e_2 \in E: e_1 \cup e_2 \supseteq \{v\} \vee \phi(e_1) = \phi(e_2), \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A.11})$$

Edge Elimination Polynomial $\xi(G, x, y, z)$

Literature: [3; 4; 5; 6; 7; 76; 77; 78; 138; 139].

Definition A.11 (Equation (13) in [4]). Let $G = (V, E)$, G^1, G^2 be graphs and $e \in E$ an edge of G . The *edge elimination polynomial* $\xi(G) = \xi(G, x, y, z)$ is defined as

$$\xi(G) = \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{+e}), \quad (\text{A.12})$$

$$\xi(G^1 \cup G^2) = \xi(G^1) \cdot \xi(G^2), \quad (\text{A.13})$$

$$\xi(K_1) = x. \quad (\text{A.14})$$

Extended Negami Polynomial $\tilde{f}(G, t, x, y, z)$

Literature: [105; 106].

Definition A.12 (Page 327 of [105]). Let $G = (V, E)$ be a graph. The *extended Negami polynomial* $\tilde{f}(G, t, x, y, z)$ is defined (for $t \in \mathbb{N}$) as

$$\tilde{f}(G, t, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, t\}} \prod_{e \in E} w(e), \quad (\text{A.15})$$

where

$$w(e) = \begin{cases} x + y & \text{if } \forall v \in e: \phi(v) = 1, \\ z + y & \text{if } \exists c \neq 1 \forall v \in e: \phi(v) = c, \\ y & \text{if } \nexists c \forall v \in e: \phi(v) = c. \end{cases} \quad (\text{A.16})$$

Extended Subgraph Counting Polynomial $H'(G, v, x, y, z)$

Definition A.13 (Definition 4.13). Let $G = (V, E)$ be a hypergraph. The *extended subgraph counting polynomial* $H'(G, v, x, y, z)$ is defined as

$$H'(G, v, x, y, z) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|} z^{c(H)}. \quad (\text{A.17})$$

Generalized Subgraph Counting Polynomial $F(G, v, x, y, z)$

Definition A.14 (Definition 5.1). Let $G = (V, E)$ be a hypergraph. The *generalized subgraph counting polynomial* $F(G, v, x, y, z)$ is defined as

$$F(G, v, x, y, z) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|} z^{|E(G[W])|}. \quad (\text{A.18})$$

Homomorphism Polynomial $H(G, k, x, y, z)$

Literature: [59].

Definition A.15 (Page 1044 in [59]). Let $G = (V, E)$ be a graph. The *homomorphism polynomial* $H(G, k, x, y, z)$ is defined (for $k, x, y, z \in \mathbb{N}$) as

$$H(G, k, x, y, z) = \text{hom}(G, H_{k,x,y,z}), \quad (\text{A.19})$$

where the graph K_k^l is a complete graph on k vertices with l loops attached at each vertex and the graph $H_{k,x,y,z}$ arises by the join of a K_k^z with the disjoint union of y copies of K_x^z .

Hyperedge Elimination Polynomial $\xi(G, x, y, z)$

Literature: [150].

Definition A.16 (Defintion 1 in [150]). Let $G = (V, E)$ be a hypergraph. The *hyperedge elimination polynomial* $\xi(G, x, y, z)$ is defined as

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(G \setminus (A \cup B)) - k(G[B])} y^{|A| + |B| - k(G[B])} z^{k(G[B])}, \quad (\text{A.20})$$

where $(A \sqcup B) \subseteq E$ is used for the summation over pairs of edge subsets $(A, B): A, B \subseteq E$, such that the set of vertices incident to the edges of A and B are disjoint: $\bigcup_{e \in A} e \cap \bigcup_{e \in B} e = \emptyset$.

Independence polynomial $I(G, x)$

Literature: [31; 68; 94; 95].

Definition A.17. Let $G = (V, E)$ be a graph. The *independence polynomial* $I(G, x)$ is defined as

$$I(G, x) = \sum_{W \subseteq V} [W \text{ is independent set}] x^{|W|}. \quad (\text{A.21})$$

Matching Polynomial $M(G, x, y)$

Literature: [56; 64; 71; 80; 93].

Definition A.18. Let $G = (V, E)$ be a graph. The *matching polynomial* $M(G, x, y)$ is defined as

$$M(G, x, y) = \sum_{A \subseteq E} [A \text{ is matching}] x^{|V| - |\bigcup_{e \in A} e|} y^{|A|}. \quad (\text{A.22})$$

Negami Polynomial $f(G, t, x, y)$

Literature: [63; 104; 105; 106; 109].

Definition A.19 (Page 601 in [104]). Let $G = (V, E)$ be a graph. The *Negami polynomial* $f(G, t, x, y)$ is defined as

$$f(G, t, x, y) = x \cdot f(G_{/e}, t, x, y) + y \cdot f(G_{-e}, t, x, y), \quad (\text{A.23})$$

$$f(E_n, t, x, y) = t^n. \quad (\text{A.24})$$

Partition Polynomial $Q(G, x)$

Literature: [127].

Definition A.20. Let $G = (V, E)$ be a graph. The *partition polynomial* $Q(G, x)$ is defined as

$$Q(G, x) = \sum_{\pi \in \Pi_c(G)} x^{|\pi|}. \quad (\text{A.25})$$

Potts Model $Z(G, x, y)$ **Literature:** [128; 129; 149; 156].**Definition A.21** (Equation (1.1) in [129]). Let $G = (V, E)$. The *Potts model* $Z(G, x, y)$ is defined as

$$Z(G, x, y) = \sum_{A \subseteq E} x^{k(G\langle A \rangle)} y^{|A|}. \quad (\text{A.26})$$

Rank Polynomial $R(G, x, y)$ **Literature:** [142; 143].**Definition A.22** (Equation (5) in [142]). Let $G = (V, E)$ be a graph. The *rank polynomial* $R(G, x, y)$ is defined as

$$R(G, x, y) = \sum_{A \subseteq E} x^{r(G\langle A \rangle)} y^{|A| - r(G\langle A \rangle)}, \quad (\text{A.27})$$

where $r(G\langle A \rangle) = |V| - k(G\langle A \rangle)$.**Rank-generating Function** $S(G, x, y)$ **Literature:** [25; 33; 147].**Definition A.23** (Section 6.2 in [33]). Let $G = (V, E)$ be a graph. The *rank-generating function* $S(G, x, y)$ is defined as

$$S(G, x, y) = \sum_{A \subseteq E} x^{r(E) - r(A)} y^{|A| - r(A)}, \quad (\text{A.28})$$

where $r(A)$ is the rank of the set A in the cycle matroid of G , i.e. $r(A) = |V| - k(G\langle A \rangle)$.**Reliability Polynomial** $R(G, p)$ **Literature:** [23; 36; 40; 54; 65; 112; 149].**Definition A.24** (Equation (3.4.1) in [149]). Let $G = (V, E)$. The *reliability polynomial* $R(G, p)$ is defined as

$$R(G, p) = \sum_{A \subseteq E} [k(G\langle A \rangle) = 1] p^{|A|} (1 - p)^{|E \setminus A|}. \quad (\text{A.29})$$

Subgraph Component Polynomial $Q(G, x, y)$ **Literature:** [7; 59; 134].**Definition A.25** (Section 1.1 in [134]). Let $G = (V, E)$ be a graph. The *subgraph component polynomial* $Q(G, v, x)$ is defined as

$$Q(G, v, x) = \sum_{W \subseteq V} v^{|W|} x^{k(G[W])}. \quad (\text{A.30})$$

Subgraph Counting Polynomial $H(G, v, x, y)$

Literature: [138].

Definition A.26 (Definition 4.8). Let $G = (V, E)$ be a hypergraph. The *subgraph counting polynomial* $H(G, v, x, y)$ is defined as

$$H(G, v, x, y) = \sum_{H=(W,F) \subseteq G} v^{|W|} x^{k(H)} y^{|F|}. \quad (\text{A.31})$$

Subgraph Enumerating Polynomial $P(G, u, v, p)$

Literature: [29].

Definition A.27 (Equation (1) in [29]). Let $G = (V, E)$ be a graph. The *subgraph enumerating polynomial* $P(G, u, v, p)$ is defined as

$$P(G, u, v, p) = \sum_{A \subseteq E} u^{\bar{i}(G \langle A \rangle)} v^{|A|} p^{k(G \langle A \rangle)}, \quad (\text{A.32})$$

where $\bar{i}(G)$ is the number of non-isolated vertices in G .

Trivariate Chromatic Polynomial $\tilde{P}(G, x, y, z)$

Literature: [59; 138].

Definition A.28 (Definition 4.19). Let $G = (V, E)$ be a hypergraph. The *trivariate chromatic polynomial* $\tilde{P}(G, x, y, z)$ is defined (for $x, y \in \mathbb{N}$) as

$$\tilde{P}(G, x, y, z) = \sum_{\phi: V \rightarrow \{1, \dots, x\}} \prod_{\substack{e \in E \\ \exists c \leq y \forall v \in e: \phi(v)=c}} z. \quad (\text{A.33})$$

Trivariate Chromatic Polynomial by White $P(G, p, q, t)$

Literature: [150].

Definition A.29 (Section 6 in [150]). Let $G = (V, E)$ be a hypergraph. The *trivariate chromatic polynomial by White* $P(G, p, q, t)$ is defined (for $x, y \in \mathbb{N}$) as

$$P(G, p, q, t) = \sum_{\phi: V \rightarrow \{1, \dots, q\}} \prod_{\substack{e \in E \\ \exists c \leq p \forall v \in e: \phi(v)=c}} t. \quad (\text{A.34})$$

Tutte Polynomial $T(G, x, y)$

Literature: [1; 2; 8; 12; 21; 26; 32; 33; 35; 39; 41; 42; 45; 54; 61; 66; 74; 84; 87; 88; 98; 111; 113; 129; 135; 141; 144; 147; 148].

Definition A.30 (Section 3 in [141]). Let $G = (V, E)$ be a graph with a linear order $<$ on the edge set E . The *Tutte polynomial* $T(G, x, y)$ is defined as

$$T(G, x, y) = \sum_{F \in \mathcal{F}(G)} x^{i(F, G, <)} y^{e(F, G, <)}. \quad (\text{A.35})$$

U-polynomial $U(G, X, y)$ **Literature:** [107].**Definition A.31** (Proposition 5.1 in [107]). Let $G = (V, E)$ be a graph. The *U-polynomial* $U(G, X, y)$ is defined as

$$U(G, X, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{k_i(G \langle A \rangle)} (y - 1)^{|A| - |V| + k(G \langle A \rangle)}, \quad (\text{A.36})$$

where $k_i(G)$ is the number of connected components of G with exactly i vertices.**Vertex-cover Polynomial** $\Psi(G, x)$ **Literature:** [51].**Definition A.32.** Let $G = (V, E)$ be a graph. The *vertex-cover polynomial* $\Psi(G, x)$ is defined as

$$\Psi(G, x) = \sum_{W \subseteq V} [W \text{ is vertex-cover}] x^{|W|}, \quad (\text{A.37})$$

where a vertex subset $W \subseteq V$ is a vertex-cover in G , if for each edge $e \in E$ there is a vertex incident to e in W , i.e. $e \cap W \neq \emptyset$ for all edges $e \in E$.**Wiener Polynomial** $W(G, q)$ **Literature:** [46; 53; 79; 80; 83; 102; 120; 121; 154; 155; 157; 158].**Definition A.33** (Equation (1) of [120]). Let $G = (V, E)$ be a connected graph. The *Wiener polynomial* $W(G, q)$ is defined as

$$W(G, q) = \sum_{\{u, v\} \in \binom{V}{2}} q^{d(G, u, v)}, \quad (\text{A.38})$$

where $d(G, u, v)$ is the distance of the vertices u and v in G .

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